

Signature quantization and representations of compact Lie groups

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We discuss some applications of signature quantization to the representation theory of compact Lie groups. In particular, we prove signature analogues of the Kostant formula for weight multiplicities and the Steinberg formula for tensor product multiplicities. Using symmetric functions, we also find, for type A, analogues of the Weyl branching rule and the Gel'fand–Tsetlin theorem.

The results described in this article are closely related to an article of Guillemin *et al.* (1) on signature quantization.

A symplectic manifold (M, ω) is *prequantizable* if the cohomology class of ω is an integral class, i.e., it is in the image of the map $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$. This assumption implies the existence of a *prequantum structure* on M : a line bundle, \mathbb{L} , and a connection, ∇ , such that $\text{curv}(\nabla) = \omega$. If g is a Riemannian metric compatible with ω , then, from g and ω , one gets an elliptic operator $\not{D}_C: S^+ \rightarrow S^-$, the *spin- \mathbb{C} Dirac operator*, and, by twisting this operator with \mathbb{L} , an operator $\not{D}_C^{\mathbb{L}}: S^+ \otimes \mathbb{L} \rightarrow S^- \otimes \mathbb{L}$. If M is compact one can “quantize” it by associating with it the virtual vector space

$$Q(M) = \text{Index } \not{D}_C^{\mathbb{L}}. \quad [1]$$

Moreover if G is a compact Lie group and τ is a Hamiltonian action of G on M , one gets from τ a representation of G on $Q(M)$ that is well defined up to isomorphism (independent of the choice of g).

The results described in this article are closely related to two theorems in ref. 1. In this article the authors study the signature analogue of spin- \mathbb{C} quantization; i.e., they define the virtual vector space (Eq. 1) by replacing \not{D}_C with the signature operator \not{D}_{sig} and prove signature versions of a number of standard theorems about quantized symplectic manifolds. The two theorems we will be concerned with in this article are the following.

1. Let $G = (S^1)^n$ and let M be a $2n$ -dimensional toric variety with moment polytope $\Delta \subseteq \mathbb{R}^n$. Then, for spin- \mathbb{C} quantization, the weights of the representation of G on $Q(M)$ are the lattice points, $\beta \in \Delta \cap \mathbb{Z}^n$, and each weight occurs with multiplicity 1. For signature quantization the weights are the same; however, the weight β occurs with multiplicity 2^n if β lies in $\text{Int}(\Delta)$, with multiplicity 2^{n-1} if it lies on a facet, and, in general, with multiplicity 2^{n-i} if it lies on i facets. Further details can be found in the work of Agapito (2).
2. Let G be a compact simply connected Lie group, λ a dominant weight, and $O_\lambda = M$ the coadjoint orbit of G through λ . In the spin- \mathbb{C} theory, the representation of G on $Q(M)$ is the unique irreducible representation V_λ of G with highest weight λ ; however, in the signature theory, it is the representation

$$\tilde{V}_\lambda = V_{\lambda-\rho} \otimes V_\rho, \quad [2]$$

where ρ is half the sum of the positive roots. (This is modulo the proviso that $\lambda - \rho$ be dominant.)

Ref. 1 also contains a signature version of the Kostant multiplicity formula. We recall that the Kostant multiplicity formula computes the multiplicity with which a weight, μ , of T occurs in V_λ by the formula

$$\sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K(\sigma(\lambda + \rho) - (\mu + \rho)), \quad [3]$$

where \mathcal{W} is the Weyl group, $|\sigma|$ is the length of σ in \mathcal{W} , and K , the *Kostant partition function* (described below in *Definition 1*). The signature version of the Kostant multiplicity formula computes the multiplicity $\tilde{m}_\lambda(\mu)$ with which the weight μ appears in \tilde{V}_λ by a similar formula,

$$\tilde{m}_\lambda(\mu) = \sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K_2(\sigma(\lambda) - \mu), \quad [4]$$

where K_2 is the $q = 2$ specialization of a new q analogue of the Kostant partition function, described below.

Our initial goal in writing this article was to give a purely algebraic derivation of this result; however, we noticed that there are \tilde{V}_λ analogues of a number of other basic formulas in the representation theory of compact semisimple Lie groups, in particular, an analogue of the Steinberg formula and, for $\text{GL}_k\mathbb{C}$, analogues of the Weyl branching rule and the Gel'fand–Tsetlin theorem. Some of the proofs are sketched but details can be found in ref. 3.

The Kostant Partition Function and Its q Analogues

We start by introducing the Kostant partition function.

Definition 1: The Kostant partition function for a root system Φ , given a choice of positive roots Φ_+ , is the function

$$K(\mu) = \left| \left\{ (k_\alpha)_{\alpha \in \Phi_+} \in \mathbb{N}^{|\Phi_+|} : \sum_{\alpha \in \Phi_+} k_\alpha \alpha = \mu \right\} \right|; \quad [5]$$

i.e., $K(\mu)$ is the number of ways that μ can be written as a sum of positive roots (see ref. 4).

Note that $K(\mu)$ can also be computed as the number of integer points inside the polytope

$$Q_\mu = \left\{ (k_\alpha)_{\alpha \in \Phi_+} \in \mathbb{R}_{\geq 0}^{|\Phi_+|} : \sum_{\alpha \in \Phi_+} k_\alpha \alpha = \mu \right\}. \quad [6]$$

We can write down a generating function for the $K(\mu)$ that is very similar to Euler's generating function for the number of partitions (see ref. 4, section 25.2):

$$\sum_{\mu} K(\mu) e^\mu = \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^\alpha}. \quad [7]$$

The classical q analogue $\hat{K}_q(\mu)$ of $K(\mu)$, according to Lusztig (5), keeps track of how many times the roots appear:

$$\hat{K}_q(\mu) = \sum_{(k_\alpha)_{\alpha \in \Phi_+}} q^{\sum k_\alpha}, \quad [8]$$

corresponding to the generating function

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$$\sum_{\mu} \hat{K}_q(\mu) e^{\mu} = \prod_{\alpha \in \Phi_+} \left(\sum_{m \geq 0} q^m e^{m\alpha} \right) = \prod_{\alpha \in \Phi_+} \frac{1}{1 - qe^{\alpha}}. \quad [9]$$

The q analogue $K_q(\mu)$ that interests us here is the one that counts the integer points of Q_{μ} according to how many k_{α} values are nonzero:

$$K_q(\mu) = \sum_{(k_{\alpha})_{\alpha \in Q_{\mu}}} q^{|\{k_{\alpha} > 0\}|}. \quad [10]$$

In terms of generating functions, this translates to

$$\sum_{\mu} K_q(\mu) e^{\mu} = \prod_{\alpha \in \Phi_+} \left(1 + q \sum_{m \geq 1} e^{m\alpha} \right) = \prod_{\alpha \in \Phi_+} \frac{1 + (q-1)e^{\alpha}}{1 - e^{\alpha}}. \quad [11]$$

The Representations $\tilde{V}_{\lambda} = V_{\lambda-\rho} \otimes V_{\rho}$

We are working in the context of a complex semisimple Lie algebra \mathfrak{g} with root system Φ , choice of positive roots Φ_+ , and Weyl group \mathcal{W} ; ρ is half the sum of the positive roots (or the sum of the fundamental weights). For a dominant weight λ , we denote by V_{λ} the irreducible representation of \mathfrak{g} with highest weight λ . We will call a weight λ *strictly dominant* if $\lambda - \rho$ is dominant. We will use the notation Λ^+ for the set of dominant weights, and Λ_S^+ for the set of strictly dominant weights. For a strictly dominant weight, we define the representation

$$\tilde{V}_{\lambda} = V_{\lambda-\rho} \otimes V_{\rho} \quad [12]$$

and its character

$$\tilde{\chi}_{\lambda} = \chi_{V_{\lambda-\rho} \otimes V_{\rho}} = \chi_{\lambda-\rho} \cdot \chi_{\rho}. \quad [13]$$

The following theorem of Guillemin *et al.* (1) provides a formula for the multiplicities of the weights in the weight space decomposition of \tilde{V}_{λ} . This formula is very similar to the Kostant multiplicity formula (Eq. 3), but it uses the $q = 2$ specialization of the q analogue of the Kostant partition function $K_q(\mu)$ introduced above, instead of the usual Kostant partition function. The formula for the \tilde{V}_{λ} multiplicities further distinguishes itself from the Kostant formula by being free of the ρ factors.

An Analogue of the Kostant Multiplicity Formula for the \tilde{V}_{λ}

Theorem 1 [Guillemin–Sternberg–Weitsman (1)]. *Let λ be a strictly dominant weight. Then the multiplicity of the weight ν in the tensor product $\tilde{V}_{\lambda} = V_{\lambda-\rho} \otimes V_{\rho}$ is given by*

$$\tilde{m}_{\lambda}(\nu) = \dim(\tilde{V}_{\lambda})_{\nu} = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} K_2(\omega(\lambda) - \nu), \quad [14]$$

where $|\omega|$ is the length of ω in the Weyl group.

Proof: We give a simple proof here using the Weyl character formula. This formula expresses the character χ_{λ} of V_{λ} as the quotient

$$\chi_{\lambda} = \frac{A_{\lambda+\rho}}{A_{\rho}}, \quad [15]$$

where $A_{\mu} = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\mu)}$. For ρ , we get the nice expression (ref. 4, lemma 24.3)

$$A_{\rho} = \prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2}) = e^{\rho} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}), \quad [16]$$

which means, in particular, that we get

$$\chi_{\rho} = \frac{A_{2\rho}}{A_{\rho}} = \frac{e^{2\rho} \prod_{\alpha \in \Phi_+} (1 - e^{-2\alpha})}{e^{\rho} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha})} = e^{\rho} \prod_{\alpha \in \Phi_+} (1 + e^{-\alpha}). \quad [17]$$

Thus, for λ strictly dominant,

$$\begin{aligned} \tilde{\chi}_{\lambda} &= \chi_{\lambda-\rho} \cdot \chi_{\rho} = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \\ &= \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \sum_{\mu} K_2(\mu) e^{-\mu} \\ &= \sum_{\mu} \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} K_2(\mu) e^{\omega(\lambda) - \mu}. \end{aligned} \quad [18]$$

Extracting the coefficient of e^{ν} on both sides gives Eq. 14.

The next step will be to use a formula due to Atiyah and Bott (6, 7) for the characters of the V_{λ} and \tilde{V}_{λ} to break down \tilde{V}_{λ} into its irreducible components and find their multiplicities. The Atiyah–Bott formula (6, 7) gives the character of V_{μ} as

$$\chi_{\mu} = \sum_{\omega \in \mathcal{W}} e^{\omega(\mu)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}}. \quad [20]$$

Remark 1: We can deduce this formula from the Weyl character formula (Eq. 15) by first observing that

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\omega(\alpha)}) = (-1)^{|\omega|} e^{\omega \sum \{\alpha \in \Phi_+ : \omega(\alpha) \in \Phi_-\}} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}). \quad [21]$$

Also,

$$\rho - \omega(\rho) = \sum \{\alpha \in \Phi_+ : \omega(\alpha) \in \Phi_-\}. \quad [22]$$

Combining Eq. 21 with Eq. 22 gives

$$\prod_{\alpha \in \Phi_+} (1 - e^{-\omega(\alpha)}) = (-1)^{|\omega|} e^{\rho - \omega(\rho)} \prod_{\alpha \in \Phi_+} (1 - e^{-\alpha}), \quad [23]$$

and we can translate Weyl’s character formula into the Atiyah–Bott formula using this equation.

For any $\omega \in \mathcal{W}$,

$$\chi_{\rho} = e^{\rho} \prod_{\alpha \in \Phi_+} (1 + e^{-\alpha}) = e^{\omega(\rho)} \prod_{\alpha \in \Phi_+} (1 + e^{-\omega(\alpha)}), \quad [24]$$

since characters are invariant under the Weyl group action. Using this and the Atiyah–Bott formula, we can write

$$\begin{aligned} \tilde{\chi}_{\lambda} &= \chi_{\lambda-\rho} \cdot \chi_{\rho} = \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\omega(\alpha)}}{1 - e^{-\omega(\alpha)}} \\ &= \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}} \sum_{I \subseteq \Phi_+} e^{-\omega(\alpha_I)}, \end{aligned} \quad [25]$$

where, as before, $\alpha_I = \sum_{\alpha \in I} \alpha$. This gives

$$\tilde{\chi}_{\lambda} = \sum_{I \subseteq \Phi_+} \left(\sum_{\omega \in \mathcal{W}} e^{\omega(\lambda - \alpha_I)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}} \right). \quad [26]$$

Letting $\lambda_I = \lambda - \alpha_I$, we observe that if λ_I is dominant, the Atiyah–Bott formula tells us that

$$\sum_{\omega \in \mathcal{W}} e^{\omega(\lambda - \alpha_I)} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\omega(\alpha)}} \quad [27]$$

is the character χ_{λ_I} of the irreducible representation V_{λ_I} , so that

$$\tilde{\chi}_\lambda = \sum_{I \subseteq \Phi_+} \chi_{\lambda_I} \quad \text{and} \quad \tilde{V}_\lambda = V_{\lambda - \rho} \otimes V_\rho = \bigoplus_{I \subseteq \Phi_+} V_{\lambda_I} \quad [28]$$

if all the λ_I are dominant.

Alternatively, we can obtain Eq. 25 from Eq. 18 by observing that for $\omega \in \mathcal{W}$

$$\omega \left(\prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \right) = \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\omega(\alpha)}}{1 - e^{-\omega(\alpha)}} = (-1)^{|\omega|} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}}.$$

Finally, since α_I and $\alpha_{I'}$ can be equal for different subsets I and I' , certain highest weights appear multiple times in the above sums. For the weight $\mu = \lambda_I = \lambda - \alpha_I$, we will get V_μ as many times as we can write $\alpha_I = \lambda - \mu$ as a sum of positive roots, where each positive root appears at most once. Hence

$$\tilde{V}_\lambda = \sum_{\mu} P(\lambda - \mu) V_\mu, \quad [29]$$

where the sum is over all μ such that $\mu = \lambda_I$ for some I , and $P(\nu)$ is given by

$$\sum_{\nu} P(\nu) e^\nu = \prod_{\alpha \in \Phi_+} (1 + e^\alpha). \quad [30]$$

Remark 2: David Vogan pointed out to us that this decomposition is well known and can be deduced from the Steinberg formula. For type A_n , the number of distinct μ 's in the above sum is the number of forests of labeled unrooted trees on $n + 1$ vertices (8, 9).

A Tensor Product Formula for the \tilde{V}_λ

We will derive here an analogue of the Steinberg formula for the \tilde{V}_λ . Given two representations \tilde{V}_λ and \tilde{V}_μ , the problem is to determine whether their tensor product $\tilde{V}_\lambda \otimes \tilde{V}_\mu$ can be decomposed in terms of \tilde{V}_ν 's. This is readily seen to be the case, as

$$\begin{aligned} \tilde{V}_\lambda \otimes \tilde{V}_\mu &= (V_{\lambda - \rho} \otimes V_\rho) \otimes (V_{\mu - \rho} \otimes V_\rho) \\ &= (V_{\lambda - \rho} \otimes V_\rho \otimes V_{\mu - \rho}) \otimes V_\rho. \end{aligned} \quad [31]$$

Breaking up $V_{\lambda - \rho} \otimes V_\rho \otimes V_{\mu - \rho}$ into irreducibles V_γ and tensoring each factor with V_ρ yields factors $V_\gamma \otimes V_\rho = \tilde{V}_{\gamma + \rho}$. Thus, for strictly dominant weights λ and μ , we can write

$$\tilde{V}_\lambda \otimes \tilde{V}_\mu = \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \tilde{V}_\nu \quad [32]$$

for some nonnegative integers $\tilde{N}_{\lambda\mu}^\nu$.

Theorem 2. For λ , μ and ν strictly dominant weights, the tensor product multiplicity $\tilde{N}_{\lambda\mu}^\nu$ of \tilde{V}_ν in $\tilde{V}_\lambda \otimes \tilde{V}_\mu$ is given by

$$\tilde{N}_{\lambda\mu}^\nu = \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\omega(\lambda) + \sigma(\mu) - \nu). \quad [33]$$

Proof: Starting from the equation $\tilde{V}_\lambda \otimes \tilde{V}_\mu = \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \tilde{V}_\nu$, we can use Eq. 18 to write

$$\begin{aligned} &\sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}} \cdot \tilde{\chi}_\mu \\ &= \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} e^{\tau(\nu)} \prod_{\alpha \in \Phi_+} \frac{1 + e^{-\alpha}}{1 - e^{-\alpha}}. \end{aligned}$$

Canceling terms and using Theorem 1 to write down the character $\tilde{\chi}_\mu$ yields

$$\begin{aligned} &\sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda)} \cdot \sum_{\beta \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\sigma|} K_2(\sigma(\mu) - \beta) e^\beta \\ &= \sum_{\nu \in \Lambda_S^+} \tilde{N}_{\lambda\mu}^\nu \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} e^{\tau(\nu)} \\ &\sum_{\beta \in \mathcal{W}} \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega| + |\sigma|} K_2(\sigma(\mu) - \beta) e^{\omega(\lambda) + \beta} \\ &= \sum_{\nu \in \Lambda_S^+} \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^\nu e^{\tau(\nu)} \end{aligned}$$

Substituting $\gamma = \omega(\lambda) + \beta$ on the left-hand side and $\gamma = \tau(\nu)$ on the right-hand side gives

$$\begin{aligned} &\sum_{\gamma \in \mathcal{W}} \sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\sigma(\mu) + \omega(\lambda) - \gamma) e^\gamma \\ &= \sum_{\substack{\gamma \text{ conjugate} \\ \text{to a strictly} \\ \text{dominant weight}}} \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)} e^\gamma, \end{aligned}$$

and extracting the coefficient of e^γ on both sides yields

$$\sum_{\omega \in \mathcal{W}} \sum_{\sigma \in \mathcal{W}} (-1)^{|\omega\sigma|} K_2(\sigma(\mu) + \omega(\lambda) - \gamma) = \sum_{\tau \in \mathcal{W}} (-1)^{|\tau|} \tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)}. \quad [34]$$

Now, since $\tilde{N}_{\lambda\mu}^{\tau^{-1}(\gamma)}$ vanishes unless $\tau^{-1}(\gamma)$ is strictly dominant, all the terms in the sum on the right-hand side vanish except for the one where τ is the identity (i.e. the term where $\gamma = \nu$), and we get the result.

If we denote by $N_{\lambda\mu}^\nu$ the multiplicities of the irreducible representations V_ν in the tensor product $V_\lambda \otimes V_\mu$, defined by

$$V_\lambda \otimes V_\mu = \sum_{\nu \in \Lambda^+} N_{\lambda\mu}^\nu V_\nu, \quad [35]$$

then we can write down the tensor product multiplicities $\tilde{N}_{\lambda\mu}^\nu$ for the decomposition of $\tilde{V}_\lambda \otimes \tilde{V}_\mu$ into \tilde{V}_ν 's in terms of the $N_{\lambda\mu}^\nu$ as follows:

$$\begin{aligned} \tilde{V}_\lambda \otimes \tilde{V}_\mu &= V_{\lambda - \rho} \otimes V_\rho \otimes V_{\mu - \rho} \otimes V_\rho \\ &= \left(\left(\sum_{\beta \in \Lambda^+} N_{\lambda - \rho, \rho}^\beta V_\beta \right) \otimes V_{\mu - \rho} \right) \otimes V_\rho \\ &= \left(\sum_{\beta \in \Lambda^+} \sum_{\gamma \in \Lambda^+} N_{\lambda - \rho, \rho}^\beta N_{\beta, \mu - \rho}^\gamma V_\gamma \right) \otimes V_\rho \\ &= \sum_{\beta \in \Lambda^+} \sum_{\gamma \in \Lambda^+} N_{\lambda - \rho, \rho}^\beta N_{\beta, \mu - \rho}^\gamma \tilde{V}_{\gamma + \rho} \\ &= \sum_{\nu \in \Lambda_S^+} \sum_{\beta \in \Lambda^+} N_{\lambda - \rho, \rho}^\beta N_{\beta, \mu - \rho}^{\nu - \rho} \tilde{V}_\nu, \end{aligned}$$

so that for strictly dominant ν ,

$$\tilde{N}_{\lambda\mu}^{\nu} = \sum_{\beta \in \Lambda^+} N_{\lambda-\rho, \beta}^{\nu} N_{\beta, \mu-\rho}^{\nu-\rho}. \quad [36]$$

Remark 3: In type A , there is a combinatorial interpretation for the coefficients $N_{\lambda\mu}^{\nu}$ in terms of shifted Young tableaux: they are given by a shifted analogue of the Littlewood–Richardson rule (see ref. 10).

Links with Symmetric Functions in Type A

As for the weight multiplicities and Clebsch–Gordan coefficients, a link exists between the character products $\tilde{\chi}_{\lambda} = \chi_{\lambda-\rho} \cdot \chi_{\rho}$ and symmetric functions in type A , again in terms of Schur functions.

The character of the irreducible polynomial representation V_{λ} of $\mathrm{GL}_k\mathbb{C}$, where we now think of λ as a partition with k parts (allowing the empty part) is the Schur function $s_{\lambda}(x_1, \dots, x_k)$. We will call a partition *strict* if all its parts are distinct (corresponding to a strictly dominant weight). Thus, we have that, for $\mathrm{GL}_k\mathbb{C}$,

$$\tilde{\chi}_{\lambda} = \chi_{\lambda-\rho} \cdot \chi_{\rho} = s_{\lambda-\rho}(x_1, \dots, x_k) s_{\rho}(x_1, \dots, x_k), \quad [37]$$

for any strict partition λ . It is readily checked that the weight ρ corresponds to the partition $(k-1, k-2, \dots, 1, 0)$.

Remark 4: We can also write the characters of \tilde{V}_{λ} in terms of Hall–Littlewood polynomials (see ref. 11, [III, 1. and 2.]). The results of the following sections can be deduced from this link with Hall–Littlewood polynomials, but we will rather use the Schur function expression (Eq. 37) for the characters. This makes the proofs a bit more technical but avoids the heavier machinery of Hall–Littlewood polynomials.

A Branching Rule for the \tilde{V}_{λ} in Type A

We have seen that the representations \tilde{V}_{λ} behave somewhat like irreducible representations, in that tensor products of them can be broken down into direct sums of \tilde{V}_{ν} 's again and that the multiplicities in those decompositions as well as in the weight space decomposition are given by formulas very similar to those of Kostant and Steinberg in the irreducible case. The Weyl branching rule (see ref. 4, for example) describes how to restrict a representation V_{λ} from $\mathrm{GL}_k\mathbb{C}$ to $\mathrm{GL}_{k-1}\mathbb{C}$. This rule can be applied iteratively and provides a way to index one-dimensional subspaces of V_{λ} by diagrams [Gel'fand–Tsetlin diagrams (12)] that is compatible with the weight space decomposition. It is natural to ask whether the representations \tilde{V}_{λ} of $\mathrm{GL}_k\mathbb{C}$ are also well behaved under restriction, or, in another words, if there is an analogue of the Weyl branching rule for the \tilde{V}_{λ} in type A .

For two partitions $\mu = (\mu_1, \dots, \mu_m)$ and $\gamma = (\gamma_1, \dots, \gamma_{m-1})$, we say that γ *interlaces* μ , and write $\gamma \triangleleft \mu$, if

$$\mu_1 \geq \gamma_1 \geq \mu_2 \geq \gamma_2 \geq \mu_3 \geq \dots \geq \mu_{m-1} \geq \gamma_{m-1} \geq \mu_m.$$

For two such partitions μ and γ such that $\gamma \triangleleft \mu$, we define

$$\nabla(\mu, \gamma) = |\{i \in \{1, 2, \dots, m-1\} : \mu_i > \gamma_i > \mu_{i+1}\}|. \quad [38]$$

In other words, $\nabla(\mu, \gamma)$ is the number of γ_i that are wedged strictly between μ_i and μ_{i+1} .

Theorem 3. *The decomposition of the restriction of the representation \tilde{V}_{λ} of $\mathrm{GL}_k\mathbb{C}$ to $\mathrm{GL}_{k-1}\mathbb{C}$ into irreducible representations of $\mathrm{GL}_{k-1}\mathbb{C}$ is given by*

$$\mathrm{Res}_{\mathrm{GL}_{k-1}\mathbb{C}}^{\mathrm{GL}_k\mathbb{C}} \tilde{V}_{\lambda} = \bigoplus_{\substack{\nu \in \Lambda_s^+ \\ \nu \triangleleft \lambda}} 2^{\nabla(\lambda, \nu)} \tilde{V}_{\nu}. \quad [39]$$

Proof: We argue using characters and the fact that those characters can be written in terms of Schur functions. We saw above (Eq. 37) that the character of the representation \tilde{V}_{λ} of $\mathrm{GL}_k\mathbb{C}$ is the product of Schur functions $s_{\lambda-\rho}(x_1, \dots, x_k) s_{\rho}(x_1, \dots, x_k)$. We obtain the character of the restriction of \tilde{V}_{λ} to $\mathrm{GL}_{k-1}\mathbb{C}$ by setting the last variable x_k equal to 1. Using well known identities on Schur functions (see ref. 13, section 7.15, for example), we have that

$$s_{\lambda}(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda} s_{\mu}(x_1, \dots, x_{k-1}) \quad [40]$$

and

$$s_{\rho}(x_1, \dots, x_k) = \prod_{1 \leq i < j \leq k} (x_i + x_j). \quad [41]$$

Thus,

$$\begin{aligned} s_{\lambda-\rho}(x_1, \dots, x_{k-1}, 1) s_{\rho}(x_1, \dots, x_{k-1}, 1) \\ = \sum_{\mu \triangleleft \lambda-\rho} s_{\mu}(x_1, \dots, x_{k-1}) \prod_{1 \leq i < j \leq k-1} (x_i + x_j) \prod_{i=1}^{k-1} (x_i + 1). \end{aligned} \quad [42]$$

We recognize the product $\prod_{1 \leq i < j \leq k-1} (x_i + x_j)$ as the Schur function $s_{\rho}(x_1, \dots, x_{k-1})$ (where ρ now corresponds to the partition $(k-2, k-3, \dots, 1, 0)$ with $k-1$ parts) and the product $\prod_{i=1}^{k-1} (x_i + 1)$ as the sum $(e_0 + e_1 + \dots + e_{k-1})$ of elementary symmetric functions in the variables x_1, \dots, x_{k-1} . A dual version of the Pieri rule (ref. 13, section 7.15) describes how to break down the product of a Schur function with an elementary symmetric function into Schur functions:

$$s_{\mu} e_m = \sum_{\nu} s_{\nu}, \quad [43]$$

where the sum is over all ν obtained from μ by adding a vertical strip of size m , i.e., over the ν such that $\mu \subseteq \nu$ and the skew-shape ν/μ consists of m boxes, no two of which are in the same row. As we are working in $k-1$ variables, the s_{ν} with more than $k-1$ parts vanish, so we can add the further constraint that the vertical strip be confined to the first $k-1$ rows (we will say such a vertical strip has height at most $k-1$). This gives

$$\begin{aligned} s_{\lambda-\rho}(x_1, \dots, x_{k-1}, 1) s_{\rho}(x_1, \dots, x_{k-1}, 1) \\ = \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} s_{\nu}(x_1, \dots, x_{k-1}) s_{\rho}(x_1, \dots, x_{k-1}) \\ \tilde{\chi}_{\lambda}(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} \tilde{\chi}_{\nu+\rho}(x_1, \dots, x_{k-1}), \end{aligned} \quad [44]$$

where the sum is over all the ν that can be obtained from μ by adding a vertical strip of size and height at most $k-1$. We can rewrite this as

$$\tilde{\chi}_{\lambda}(x_1, \dots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda-\rho} \sum_{\nu} \tilde{\chi}_{\nu}(x_1, \dots, x_{k-1}), \quad [45]$$

where the sum is over all strict partitions ν such that $\nu - \rho$ can be obtained from μ by adding a vertical strip of size and height at most $k-1$. Since the $s_{\nu} s_{\rho}$ are linearly independent, we can lift this to the level of representations to get

$$\text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\mu \triangleleft \lambda - \rho} \bigoplus_{\nu} \tilde{V}_\nu, \quad [46]$$

with the sum over the same set of ν as before.

To compute the multiplicity of a given \tilde{V}_ν in $\text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda$, we define, for strict partitions λ and ν , $n(\lambda, \nu)$ to be the number of ways that $\nu - \rho$ can be obtained by adding a vertical strip of size and height at most $k - 1$ to some partition μ such that $\mu \triangleleft \lambda - \rho$, so that

$$\tilde{V}_\lambda = \bigoplus_{\nu \in \Lambda_S^+} n(\lambda, \nu) \tilde{V}_\nu. \quad [47]$$

Note that δ has two different meanings here: for the group $\text{GL}_k\mathbb{C}$, it corresponds to the partition $(k - 1, k - 2, \dots, 1, 0)$, while for $\text{GL}_{k-1}\mathbb{C}$, it corresponds to the partition $(k - 2, k - 3, \dots, 1, 0)$. To avoid confusion, we will denote the latter by δ' .

The condition $\mu \triangleleft \lambda - \delta$ means that

$$\begin{aligned} \lambda_1 - (k - 1) &\geq \mu_1 \geq \lambda_2 - (k - 2) \geq \mu_2 \\ &\geq \dots \geq \lambda_{k-1} - 1 \geq \mu_{k-1} \geq \lambda_k. \end{aligned}$$

Replacing μ_i by $\mu_i + \delta'_i = \mu_i + (k - 1 - i)$ gives

$$\begin{aligned} \lambda_1 - 1 &\geq \mu_1 + (k - 2) \geq \lambda_2 \\ \lambda_2 - 1 &\geq \mu_2 + (k - 1) \geq \lambda_3 \\ &\dots \quad \dots \quad \dots \\ \lambda_{k-1} - 1 &\geq \mu_{k-1} + (0) \geq \lambda_k. \end{aligned}$$

These equations mean that the i th part of $\mu' = \mu + \delta'$ is at least as large as the $(i + 1)$ th part of λ and smaller than the i th part of λ . In other words, the skew-shape λ/μ' is a horizontal strip with at least a box in each row, or equivalently $\mu' \triangleleft \lambda$ with the further constraints $\mu'_i < \lambda_i$ for all $1 \leq i \leq k - 1$. Adding a vertical strip to μ to get $\nu - \delta$ is the same as adding a vertical strip to μ' to get ν , provided that we only allow adding vertical strips to μ' that result in a strict partition. It is then clear that by adding such a vertical strip to μ' , we get a strict partition ν such that λ/ν is a horizontal strip. Conversely, it is also clear that for any strict ν such that λ/ν is a horizontal strip, there is a μ' such that ν can be obtained from μ' by adding a vertical strip. So the only summands, \tilde{V}_ν , for which $n(\lambda, \nu) \neq 0$ in the decomposition (Eq. 56) are those for which $\nu \triangleleft \lambda$.

Given such a ν , we will compute $n(\lambda, \nu)$ by constructing row by row the strict partitions $\mu' = \mu + \delta'$ from which we can obtain ν . Given v_i , there are three cases to consider for the possible μ'_i :

- $v_i = \lambda_i$. In this case, since we must have $\mu'_i < \lambda_i$, it has to be that $\mu'_i = \lambda_i - 1$ and that we have a box in row i of the vertical strip. So there is only one choice for μ'_i .
- $v_i = \lambda_{i+1}$. Then we must have $\mu_i = \lambda_{i+1} \leq \mu'_i \leq v_i$ and therefore $\mu'_i = v_i$, so we don't have a box in row i of the vertical strip. Again, there is only one choice for μ'_i in this case.
- $\lambda_i > v_i > \lambda_{i+1}$. Then we can either have $\mu'_i = v_i - 1$ and have a box from the vertical strip in row i , or have $\mu'_i = v_i$ and have no box from the vertical strip in row i . So there are two possibilities for μ'_i in this case.

We have to show that any choice of μ'_i that we make gives rise to a strict partition (by construction, it is clear that $\mu' \triangleleft \lambda$). If for some i we had $\mu'_i = \mu'_{i+1}$, then because λ_{i+1} is at least $\mu'_{i+1} + 1$, this would mean that λ_i is at least $\mu'_i + 2$, since $\lambda_i > \lambda_{i+1}$. But then λ/μ' contains two boxes in the same column: the box after box μ'_i in row i , and the box after box $\mu'_i = \mu'_{i+1}$ in row $i + 1$, which contradicts the fact that $\mu' \triangleleft \lambda$ (or equivalently, that λ/μ' is a horizontal strip). Hence we get two choices for each instance of a pattern of the form $\lambda_i > v_i > \lambda_{i+1}$. We called the number of

such instances above $\nabla(\lambda, \nu)$. Since the choices at each row are independent, we have

$$n(\lambda, \nu) = \begin{cases} 2^{\nabla(\lambda, \nu)} & \text{if } \nu \triangleleft \lambda \text{ and } \nu \in \Lambda_S^+, \\ 0 & \text{otherwise.} \end{cases} \quad [48]$$

from which the proposed expression for the branching rule follows.

Gel'fand-Tsetlin Theory for the \tilde{V}_λ

After restricting to $\text{GL}_{k-1}\mathbb{C}$, we can further restrict to $\text{GL}_{k-2}\mathbb{C}$. From now on, we will assume that all partitions are strict. We can write

$$\begin{aligned} \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda &= \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \left(\text{Res}_{\text{GL}_{k-1}\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda \right) \\ &= \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \left(\bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\mu, \nu)} \tilde{V}_\nu \right) \\ &= \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \text{Res}_{\text{GL}_{k-2}\mathbb{C}}^{\text{GL}_{k-1}\mathbb{C}} \tilde{V}_\nu \end{aligned} \quad [49]$$

$$= \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \left(\bigoplus_{\mu \triangleleft \nu} 2^{\nabla(\nu, \mu)} \tilde{V}_\mu \right) \quad [50]$$

$$= \bigoplus_{\mu \triangleleft \nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu) + \nabla(\nu, \mu)} \tilde{V}_\mu. \quad [51]$$

Denoting by $\lambda^{(m)} = \lambda_1^{(m)} \geq \dots \geq \lambda_m^{(m)} \geq 0$ the strict partitions indexing the representations \tilde{V} of $\text{GL}_m\mathbb{C}$, we can iterate the branching rule until we get to $\text{GL}_1\mathbb{C}$:

$$\text{Res}_{\text{GL}_1\mathbb{C}}^{\text{GL}_k\mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\lambda^{(1)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda} 2^{\nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \dots + \nabla(\lambda^{(2)}, \lambda^{(1)})} V_{\lambda^{(1)}}. \quad [52]$$

We will call a sequence of strict partitions of the form $\lambda^{(1)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda$ a *twisted Gel'fand-Tsetlin diagram* for λ , which can be viewed schematically as

$$\begin{array}{ccccccc} \lambda_1^{(k)} & & \lambda_2^{(k)} & & \dots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\ & \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \dots & & \lambda_{k-1}^{(k-1)} & \\ & & \ddots & & \vdots & & \ddots & & \\ & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array} \quad [53]$$

with $\lambda_j^{(k)} = \lambda_j$ and each $\lambda_j^{(i)}$ is a nonnegative integer satisfying

$$\lambda_j^{(i)} > \lambda_{j+1}^{(i)} \quad [54]$$

and

$$\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)} \quad [55]$$

for all $1 \leq j \leq i$, $1 \leq i \leq k - 1$. Let $\tilde{V}_{\mathcal{D}}$ be the subspace of \tilde{V}_λ corresponding to a twisted Gel'fand-Tsetlin diagram \mathcal{D} . This subspace has dimension $2^{\nabla(\mathcal{D})}$, where

$$\nabla(\mathcal{D}) = \nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \nabla(\lambda^{(k-1)}, \lambda^{(k-2)}) + \dots + \nabla(\lambda^{(2)}, \lambda^{(1)}). \quad [56]$$

We can also think of $\nabla(\mathcal{D})$ as the number of triangles

$$\begin{array}{ccc} \lambda_j^{(i)} & & \lambda_{j+1}^{(i)} \\ & \searrow & \nearrow \\ & \lambda_j^{(i+1)} & \end{array}$$

with strict inequalities $\lambda_j^{(i+1)} > \lambda_j^{(i)} > \lambda_{j+1}^{(i+1)}$ in the diagram \mathcal{D} .

We show here that $\tilde{V}_{\mathcal{D}}$ lies completely within the same weight space as the weight space decomposition of \tilde{V}_{λ} . We will think of the groups $\mathrm{GL}_k\mathbb{C}$ as included into one another by identifying $\mathrm{GL}_m\mathbb{C}$ with

$$\left(\begin{array}{c|c} \mathrm{GL}_m\mathbb{C} & \mathbf{0} \\ \hline \mathbf{0} & id_{k-m} \end{array} \right).$$

The maximal forces of $\mathrm{GL}_k\mathbb{C}$ is its subgroup of invertible matrices T_k , whose Lie algebra will be denoted t_k . We will consider two bases of t_k : let

$$I_m = \left(\begin{array}{c|c} id_m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right),$$

$1 \leq m \leq k$, and $J_m = I_m - I_{m-1}$ for $1 \leq m \leq k$.

Consider the element $I \in \mathfrak{gl}_m\mathbb{C}$ and a representation \tilde{V}_{μ} of $\mathrm{GL}_m\mathbb{C}$. We have the representation $\mathrm{GL}_k\mathbb{C} \rightarrow \mathfrak{gl}(V_{\mu} \otimes V_{\rho})$. For $\nu \in V_{\mu-\rho}$ and $\omega \in V_{\rho}$, we have

$$\begin{aligned} I \cdot (\nu \otimes \omega) &= (I \cdot \nu) \otimes \omega + \nu \otimes (I \cdot \omega) \\ &= \left(\left(\sum_{j=1}^m (\mu - \rho)_j \right) \nu \right) \otimes \omega + \nu \otimes \left(\left(\sum_{j=1}^m \rho_j \right) \omega \right) \\ &= \left(\sum_{j=1}^m ((\mu - \rho)_j + \rho_j) \right) \nu \otimes \omega \\ &= \left(\sum_{j=1}^m \mu_j \right) \nu \otimes \omega, \end{aligned}$$

since $V_{\mu-\rho}$ has highest weight $\mu - \rho$ and V_{ρ} has highest weight ρ . So $I \in \mathfrak{gl}_m\mathbb{C}$ gets represented as $(\sum_{j=1}^m \mu_j) I$ in \tilde{V}_{μ} . In general, for

$$\mathrm{Res}_{\mathrm{GL}_m\mathbb{C}}^{\mathrm{GL}_k\mathbb{C}} \tilde{V}_{\lambda} = \bigoplus_{\lambda^{(m)} \triangleleft \dots \triangleleft \lambda^{(k)} = \lambda} 2^{\nabla(\lambda^{(k)}, \lambda^{(k-1)}) + \dots + \nabla(\lambda^{(m+1)}, \lambda^{(m)})} \tilde{V}_{\lambda^{(m)}},$$

we will find that $I \in \mathfrak{gl}_m\mathbb{C}$ gets represented as $(\sum_{i=1}^m \lambda_i^{(m)}) I$ in $\tilde{V}_{\lambda^{(m)}}$. Therefore, in the basis I_1, \dots, I_k , the subspace $\tilde{V}_{\mathcal{D}}$ corresponding to a twisted Gel'fand–Tsetlin diagram \mathcal{D} has weight

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$$\left(\sum_{i=1}^1 \lambda_i^{(1)}, \sum_{i=1}^2 \lambda_i^{(2)}, \dots, \sum_{i=1}^k \lambda_i^{(k)} \right)$$

or

$$\left(\sum_{i=1}^1 \lambda_i^{(1)}, \sum_{i=1}^2 \lambda_i^{(2)} - \sum_{i=1}^1 \lambda_i^{(1)}, \dots, \sum_{i=1}^k \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)} \right)$$

in the usual basis J_1, \dots, J_k .

In other words, $\tilde{V}_{\mathcal{D}} \subseteq (\tilde{V}_{\lambda})_{\beta}$ if

$$\beta_m = \sum_{i=1}^m \lambda_i^{(m)} - \sum_{i=1}^{m-1} \lambda_i^{(m-1)}, \quad [57]$$

or, equivalently,

$$\beta_1 + \dots + \beta_m = \sum_{i=1}^m \lambda_i^{(m)}. \quad [58]$$

Hence twisted Gel'fand–Tsetlin diagrams for λ correspond to the same weight if all their row sums are the same. So we have proved the following analogue of the Gel'fand–Tsetlin theorem (12).

Theorem 4. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a strictly dominant weight. The dimension of the representation \tilde{V}_{λ} of $\mathrm{GL}_k\mathbb{C}$ is given by

$$\dim \tilde{V}_{\lambda} = \sum_{\mathcal{D}} 2^{\nabla(\mathcal{D})} \quad [59]$$

where the sum is over all twisted Gel'fand–Tsetlin diagrams with top row λ . Furthermore, the multiplicity $\tilde{m}_{\lambda}(\beta)$ of the weight β in \tilde{V}_{λ} is given by

$$\tilde{m}_{\lambda}(\beta) = \dim (\tilde{V}_{\lambda})_{\beta} = \sum_{\mathcal{D}} 2^{\nabla(\mathcal{D})}, \quad [60]$$

where the sum is over all twisted Gel'fand–Tsetlin diagrams with top row λ and row sums satisfying Eq. 57 (or Eq. 58).

Remark 5: We can also prove that $\tilde{V}_{\mathcal{D}}$ lies completely within a weight space of \tilde{V}_{λ} using characters and Schur function identities.

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