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TRANSITION CURVES IN THE QUASIPERIODIC MATHIEU EQUATION

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ABSTRACT

In this work we investigate the following quasiperiodic Mathieu equation:

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)\, x = 0$$

We use numerical integration and Lyapunov exponents to determine transition curves bounding regions of stability in the $\delta - \omega$ plane for fixed ϵ. In addition, we obtain approximate analytic expressions for these transition curves using two distinct methods: regular perturbations and harmonic balance. Comparison of the results of these methods with those of numerical integration shows that the perturbation method fails to converge in the neighborhood of resonant values of ω. The results obtained by harmonic balance do not display this undesirable feature.

INTRODUCTION

Mathieu's equation [5],

$$\ddot{x} + (\delta + \epsilon \cos t)\, x = 0 \qquad (1)$$

is the paradigm for problems in parametric excitation, in which an autonomous linear structure is driven by a periodic forcer, usually in a direction perpendicular to the direction of motion (e.g. the vertically forced pendulum, dynamic buckling of an elastic column, water waves in a vertically driven channel).

This work concerns a natural extension of such problems to cases in which the forcer is quasiperiodic. We investigate the following quasiperiodic Mathieu equation:

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \omega t)\, x = 0 \qquad (2)$$

For a given set of parameters δ, ω, ϵ, eq.(2) is said to be **stable** if all solutions are bounded, and **unstable** otherwise. In a previous work [3], numerical integration was used to determine which points in the $\delta - \omega$ plane were stable, for fixed ϵ. Also in [3], a perturbation method was presented to study the stability of eq.(2) for small ω and ϵ when δ was close to $\frac{1}{4}$. Eq.(2) has also been investigated by perturbation methods in [1], [2],[6]. These authors have noted the failure of perturbations for small ϵ when ω takes on resonant values due to small-divisors. We shall show how the method of harmonic balance can be used to avoid these difficulties.

NUMERICAL INTEGRATION

In order to obtain an approximate stability chart, we numerically integrated eq.(2) forward in time from arbitrarily chosen initial conditions at $t = 0$ up to $t = 20,000$. At each step we computed the amplitude $\sqrt{x(t)^2 + \dot{x}(t)^2}$ and judged a motion to be unstable if its amplitude became greater than a million times its initial value for any t between 0 and 20,000, and stable otherwise. Our results are displayed in Fig.1.

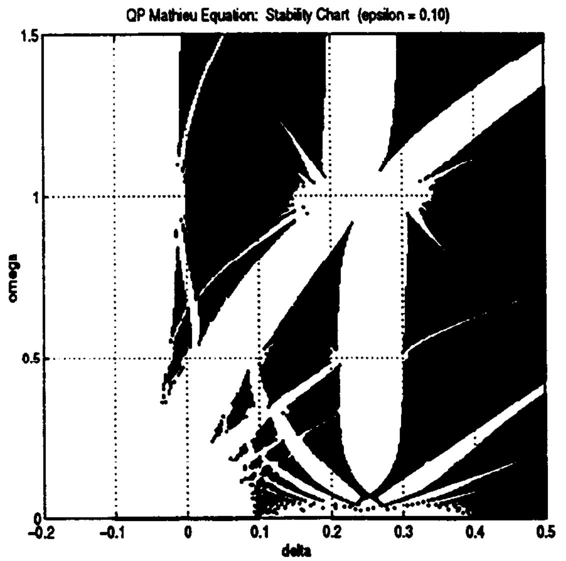

Fig.1. Stability of eq.(2) as determined directly from numerical integration. Points (δ, ω) in the blackened regions of the δ-ω parameter plane correspond to stable (bounded) solutions.

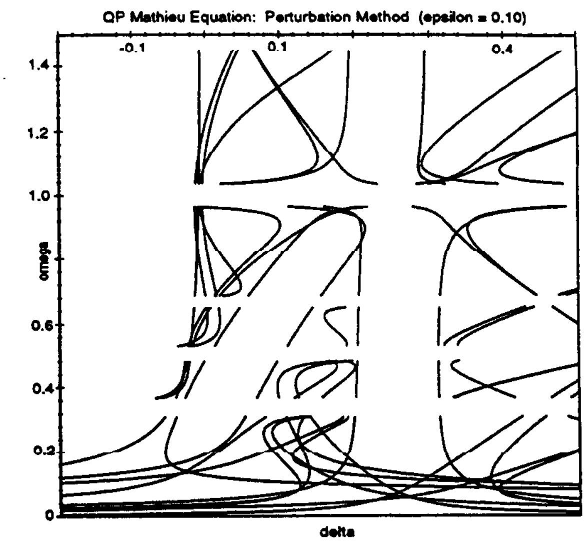

Fig.3 Transition curves of eq.(2) as determined from the perturbation method. The regions around resonant values of ω have been omitted.

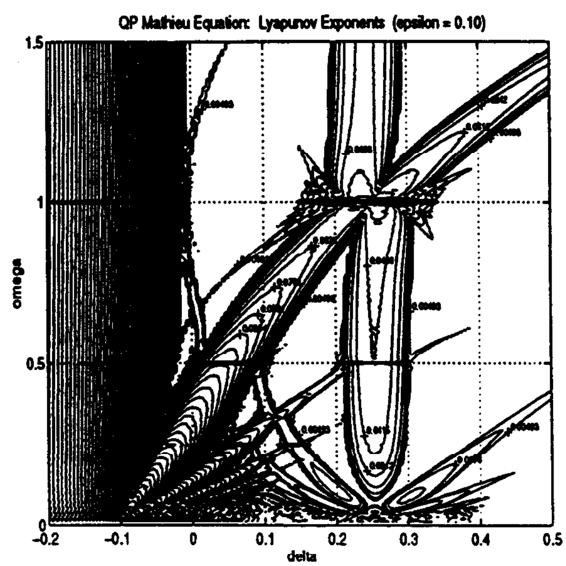

Fig.2. Contour plot of Lyapunov exponents. The level curves correspond to constant values of $\lambda > 0$ for which eq.(2) is unstable.

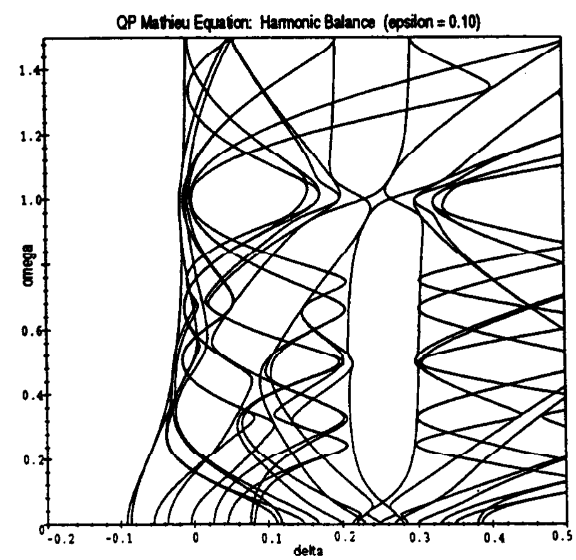

Fig.4. Transition curves of eq.(2) as determined by the method of harmonic balance for truncation order $N = 4$.

LYAPUNOV EXPONENTS

Lyapunov exponents [5] provide a second approach based on numerical integration with which we can obtain an approximate stability chart for eq.(2). The *Lyapunov exponent* of the solution $x(t)$ is defined as:

$$\lambda = \limsup_{t\to\infty} \frac{1}{t} \ln | x(t) | \qquad (3)$$

A positive Lyapunov exponent, $\lambda > 0$, corresponds to an unstable solution, and since there is no dissipation in eq.(2), stable solutions correspond to $\lambda = 0$. After numerically integrating eq.(2), we found it difficult to distinguish between small positive Lyapunov exponents and those which were truly zero. We resolved this by noting that Lyapunov exponents which are genuinely nonzero should maintain their values as the time of integration is increased. On the other hand, Lyapunov exponents that are genuinely zero will have computed approximations that tend to zero as the time of integration is increased. Our results are displayed as a contour plot in Fig.2.

PERTURBATION METHOD

In the case of Mathieu's eq.(1), regular perturbations have been used to provide approximate analytic expressions for the transition curves in the $\delta - \epsilon$ plane which separate regions of stability from regions of instability [6]. The procedure, which is valid for small values of ϵ, is based on a result from Floquet theory, namely that along a transition curve there exist solutions with period 2π or 4π. This leads to the conclusion that when $\epsilon = 0$, regions of instability ("Arnold tongues") emanate from points on the δ axis of the form:

$$\delta = \delta_0 = \frac{n^2}{4}, \qquad n = 1, 2, 3, \cdots \qquad (4)$$

Thus in the case of Mathieu's eq.(1), we set

$$\delta = \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots \qquad (5)$$

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \qquad (6)$$

where $x_0(t)$ is separately equated to each of $\sin nt/2$ and $\cos nt/2$, each of which, together with eq.(4), yields a transition curve. The pair of curves associated with a given value of n yield an approximation for the corresponding Arnold tongue. (In the case of $n = 0$, however, a single transition curve occurs corresponding to $x_0(t) = 1$.) After substituting these power series expansions into Mathieu's eq.(1), the coefficient δ_i is obtained by eliminating secular terms from the eq. on $x_i(t)$.

We shall apply a similar scheme to the QP Mathieu eq.(2). The key ansatz is the generalization of the transition points of eq.(4) to:

$$\delta_0 = \frac{(n + m\omega)^2}{4}, n = 0, 1, 2, ..., m = 0, \pm1, \pm2, ..., \qquad (7)$$

The expansions (5),(6) are substituted into the QP Mathieu eq.(2), where $x_0(t)$ is separately equated to each of $\sin(\frac{n+m\omega}{2}t)$ and $\cos(\frac{n+m\omega}{2}t)$, each of which, together with eq.(7), yields a transition curve. Once again, the coefficient δ_i is obtained by eliminating secular terms from the eq. on $x_i(t)$.

As an example of the kind of results obtained in this way (using computer algebra), take the cases $(n, m) = (1, 1)$ and $(1, -1)$:

Case $\delta_0 = \frac{1}{4}(1+\omega)^2$: $n = 1$ *and* $m = 1$

$$\delta = \frac{1}{4}(\omega + 1)^2 - \frac{1}{2}\frac{\epsilon^2\,(\omega^2 + \omega + 1)}{\omega\,(2\omega + 1)(\omega + 2)} + O(\epsilon^4) \quad (8)$$

$$\delta = \frac{1}{4}(\omega + 1)^2 + \frac{3}{2}\frac{\epsilon^2\,(\omega^2 + 3\omega + 1)}{\omega\,(2\omega + 1)(\omega + 2)} + O(\epsilon^4) \quad (9)$$

Case $\delta_0 = \frac{1}{4}(1-\omega)^2$: $n = 1$ *and* $m = -1$

$$\delta = \frac{1}{4}(1 - \omega)^2 + \frac{1}{2}\frac{\epsilon^2\,(\omega^2 - \omega + 1)}{\omega\,(2\omega - 1)(\omega - 2)} + O(\epsilon^4) \quad (10)$$

$$\delta = \frac{1}{4}(1 - \omega)^2 - \frac{3}{2}\frac{\epsilon^2\,(\omega^2 - 3\omega + 1)}{\omega\,(2\omega - 1)(\omega - 2)} + O(\epsilon^4) \quad (11)$$

Plots of transition curves generated in this way for $n = 0, 1, 2$ and $m = 0, \pm1, \pm2$, all valid to $O(\epsilon^4)$, are displayed in Fig.3. Note that the expressions for the transition curves are not valid in neighborhoods of $\omega = 0$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1, $\frac{3}{2}$, 2 and 3, since there are terms that have vanishing denominators at these resonant values, cf.eqs.(10),(11). For this reason, these portions of the transition curves are omitted in Fig.3. Additional resonances will show up at higher order truncations.

HARMONIC BALANCE

Another approach which has been used successfully in the case of Mathieu's eq.(1) is harmonic balance [6]. This method is again based on the result from Floquet theory that along a transition curve there exist solutions with period 2π or 4π, leading to the following Fourier expansion:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\frac{k}{2}t + B_k \sin\frac{k}{2}t \qquad (12)$$

Substitution of eq.(12) into eq.(1) and collecting terms (i.e. balancing harmonics) leads to an infinite set of linear, homogeneous equations for the coefficients $\{A_k, B_k\}$. For a nontrivial solution, the infinite (or Hill's) determinant of the associated coefficient matrix must vanish. By truncating this infinite system, we obtain an approximate implicit equation for the transition curves in the δ-ϵ parameter plane.

We shall apply a similar scheme to the QP Mathieu eq.(2). This time the key ansatz is the generalization of the Fourier series eq.(12) to the quasiperiodic form:

$$x(t) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} A_{nm} \cos(\frac{n+m\omega}{2}t) + B_{nm} \sin(\frac{n+m\omega}{2}t) \qquad (13)$$

In practice, the infinite sums in eq.(13) are replaced by sums from 0 to N and from $-N$ to N, respectively, a truncation which gives approximate results. Since the forcing term in eq.(2) is an even function of t, the solution space can be spanned by an even solution and an odd solution. This permits us to take first B_{nm} and then A_{nm} as zero in eq.(13), thereby reducing the size of the (truncated) determinant by half. In the former case, we substitute eq.(13) with $B_{nm} = 0$ into eq.(2) using computer algebra, then perform a trigonometric reduction and collect terms to give:

$$A_{n,m}\left(\delta - \frac{1}{4}(n+m\omega)^2\right) + \frac{\epsilon}{2}\left(A_{n+2,m} + A_{n-2,m} + A_{n,m+2} + A_{n,m-2}\right) = 0 \quad (14)$$

Eqs.(14), when truncated at the N^{th} harmonic, represent $2N^2 + 2N + 1$ simultaneous equations. E.g. in the case of $N = 4$, the matrix of coefficients has dimension 41, and the evaluation of the corresponding determinant and the generation of graphical plots took an hour using Maple on a Sun SPARCstation 10. As an example of the kind of results obtained by this method, see the Appendix where the determinant is displayed in factored form for the truncation $N = 2$. Fig.4 shows both the sine and cosine solutions for $N = 4$. Note the absence of the small-divisor problem, in contrast to the perturbation method results of Fig.3.

CONCLUSIONS

A comparison of Figs.1 - 4 reveals general agreement between the numerical and analytical approximations of the stability of the quasiperiodic Mathieu eq.(2). The method of Lyapunov exponents in Fig.2 shows greater detail than the simple numerical integration of Fig.1, although Fig.1 involved four times more computation than Fig.2! Fig.3 based on the perturbation method displays gaps where portions of the transition curves (which lie near the resonant values of ω) have been omitted, while Fig.4 generated by the method of harmonic balance does not suffer from this singular behavior. All four methods show that for small ϵ the two largest instability regions lie in the neighborhood of the curves $\delta = \frac{1}{4}$ and $\delta = \frac{\omega^2}{4}$, each of which represents a 2:1 resonance between the respective driving frequency (1 and ω) and the unforced frequency ($\sqrt{\delta}$). The analytical methods show that the thickness of these two instability regions is approximately ϵ. However it is evident both from the numerical plots as well as from the analytical results that there are many additional smaller instability regions. E.g., the point of intersection between these two largest instability regions, namely $\delta = \frac{1}{4}, \omega = 1$, is the birthplace of a number of smaller instability regions, see Figs.1 and 2. For this choice of parameters, both driving frequencies are in 2:1 resonance with the unforced frequency and we may expect the unfolding of such a degeneracy to be accompanied by a diversity of bifurcations.

Why is it that the perturbation method fails to give good results at points where the harmonic balance method works well? We offer the following explanation based on a simple model which is not directly related to the quasiperiodic Mathieu eq.(2), but which we think captures the essential phenomenon. Suppose that we were interested in a perturbation solution to a problem which had the form:

$$(\delta - \frac{1}{4})(\delta - \frac{\omega^2}{4}) = \epsilon \qquad (15)$$

This form is chosen because it is a simplified version of the kind of expression generated by the harmonic balance method. In seeking a perturbation solution we would expand δ in a power series in ϵ:

$$\delta = \delta_0 + \epsilon\delta_1 + \cdots \qquad (16)$$

Substituting eq.(16) into eq.(15) and collecting terms gives the perturbation expansion:

$$\delta = \frac{1}{4} + \frac{4}{1-\omega^2}\epsilon + \cdots \qquad (17)$$

Note that eq.(17) is singular at $\omega^2 = 1$, in a manner similar to the misbehavior of the perturbation solutions of the quasiperiodic Mathieu eq.(2), cf.eqs.(10), (11). We may say that the reason the perturbation method failed is that the assumed form of the solution, eq.(16), is inappropriate. By contrast, the implicit form of eq.(15), which is analogous to the results of the harmonic balance method, has no small-divisor difficulty.

Why does the harmonic balance method work? We offer an explanation based on restricting ω to rational values: $\omega = \frac{p}{q}$ where p and q are relatively prime positive integers. We may justify this restriction by noting that any irrational number can be approximated by a rational to any degree of accuracy. With this restriction, the quasiperiodic Mathieu eq.(2) becomes the following Hill's equation:

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos \frac{p}{q} t)\, x = 0 \qquad (18)$$

Assuming $\omega < 1$, the term $\epsilon \cos t + \epsilon \cos \frac{p}{q} t$ has period $T = 2\pi q$. According to Floquet theory, any solution $x(t)$ along the transition curves of equation (18) has minimum period T or $2T$, and hence, can be expanded in a Fourier series:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos \frac{k}{2q} t + B_k \sin \frac{k}{2q} t \qquad (19)$$

Since p and q are relatively prime, any integer k can be expressed as the linear combination $k = nq + mp$ [3]. As a result, the set of integers can be put into a one-to-one correspondence with the following set of ordered pairs of integers:

$$\mathbf{Z} \longleftrightarrow \mathcal{M}$$

where

$$\mathcal{M} \equiv \{(\mathrm{n},\mathrm{m}) \in \mathbf{Z}\times\mathbf{Z} : \ \mathrm{k} = \mathrm{nq} + \mathrm{mp},\ \mathrm{k} \in \mathbf{Z}\} \quad (20)$$

Ordered pairs that yield the same integer are identified and defined to be in the same equivalence class. Hence, the above correspondence is actually between $\mathbf{Z}$ and the set of equivalence classes. The Fourier series (19) for $x(t)$ can then be expressed as follows:

$$\begin{aligned} x(t) &= \sum_{\mathcal{M}} A_{nm} \cos \frac{nq+mp}{2q} t \ + \ B_{nm} \sin \frac{nq+mp}{2q} t \\ &= \sum_{\mathcal{M}} A_{nm} \cos \frac{n+m\omega}{2} t \ + \ B_{nm} \sin \frac{n+m\omega}{2} t \end{aligned}$$

which is in the form of the ansatz given by equation (13).

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APPENDIX

We present below the results of the harmonic balance method for truncation $N = 2$.

sine-series solution:

$$
\begin{aligned}
\det = 0 = \; & -64\left(-\omega^2 + 2\omega + 4\delta - 1\right)\left(4\delta - 1 - 2\omega - \omega^2\right) \\
& \Big(8\epsilon^3 + \left(-12\omega^2 + 48\delta - 32\right)\epsilon^2 + \left(-16\omega^2 + 2\omega^4 + 32 + 32\delta^2 - 64\delta - 16\omega^2\delta\right)\epsilon \\
& \quad + 16\omega^2 + \omega^6 + 128\delta^2 - 64\delta - 64\delta^3 - 12\omega^4\delta + 48\omega^2\delta^2 - 8\omega^4\Big) \\
& \Big(8\epsilon^3 + \left(48\delta - 32\omega^2 - 12\right)\epsilon^2 + \left(-16\omega^2 + 32\delta^2 - 64\omega^2\delta - 16\delta + 2 + 32\omega^4\right)\epsilon \\
& \quad - 8\omega^2 + 1 - 64\omega^4\delta + 48\delta^2 + 16\omega^4 - 64\delta^3 + 128\omega^2\delta^2 - 12\delta\Big) \\
& \Big(\epsilon^4 + \left(-4\omega^2 + 6\delta - 2 - 4\delta^2 - 2\omega^4 + 6\omega^2\delta\right)\epsilon^2 \\
& \quad + 4\omega^2 + 12\omega^2\delta^2 - 8\omega^4 - 4\omega^6\delta + 12\omega^4\delta^2 - 12\delta^3\omega^2 \\
& \quad + 4\omega^6 - 4\omega^2\delta + 12\delta^2 - 12\delta^3 - 4\delta - 4\omega^4\delta + 4\delta^4\Big)
\end{aligned}
$$

cosine-series solution:

$$
\begin{aligned}
\det = 0 = \; & 512\left(-16\epsilon^2 - 2\omega^2 + 1 - 8\delta - 8\omega^2\delta + 16\delta^2 + \omega^4\right) \\
& \Big(8\epsilon^3 + \left(32\omega^2 - 48\delta + 12\right)\epsilon^2 + \left(-16\omega^2 + 32\delta^2 - 64\omega^2\delta - 16\delta + 2 + 32\omega^4\right)\epsilon \\
& \quad + 64\omega^4\delta - 48\delta^2 - 16\omega^4 - 128\omega^2\delta^2 - 1 + 64\delta^3 + 8\omega^2 + 12\delta\Big) \\
& \Big(-8\epsilon^3 + \left(-12\omega^2 + 48\delta - 32\right)\epsilon^2 + \left(16\omega^2 - 32 + 16\omega^2\delta + 64\delta - 2\omega^4 - 32\delta^2\right)\epsilon \\
& \quad - 12\omega^4\delta + 128\delta^2 - 8\omega^4 + 48\omega^2\delta^2 - 64\delta - 64\delta^3 + 16\omega^2 + \omega^6\Big) \\
& \Big(\left(5\omega^4\delta - 1 + 4\delta^3 - 8\omega^2\delta^2 + \omega^2 + \omega^4 + 5\delta - \omega^6 + 2\omega^2\delta - 8\delta^2\right)\epsilon^2 \\
& \quad + 2\omega^6\delta^2 + 2\delta^2 - 2\omega^2\delta + 4\omega^4\delta + 6\delta^4\omega^2 - 6\delta^3 - 2\omega^6\delta \\
& \quad + 2\omega^2\delta^2 - 6\delta^3\omega^2 + 6\delta^4 - 2\delta^5 - 6\omega^4\delta^3 + 2\omega^4\delta^2\Big)
\end{aligned}
$$

ADDENDUM

In this section we announce two recent results which are related to eq.(2):

1. Let eq.(2) be generalized to include a phase difference between the two driving terms,

$$\ddot{x} + (\delta + \epsilon \cos t + \epsilon \cos(\omega t + \phi))\, x = 0$$

Then it turns out that the transition curves are not dependent on ϕ ! This may be shown by writing this eq. in the form

$$\ddot{x} + (\delta + \epsilon \cos y + \epsilon \cos z)\, x = 0, \quad \dot{y} = 1, \quad \dot{z} = \omega$$

For irrational ω, the flow on the $y-z$ torus is dense, and hence the phase of z relative to y cannot affect boundedness as $t \to \infty$. (Thanks to John Guckenheimer.)

2. In the large ϵ limit, eq.(2) is stable for

$$\delta > 2\epsilon >> 1.$$

This may be shown by writing eq.(2) in the form:

$$\mu^2 \ddot{x} + (k + \cos t + \cos \omega t)\, x = 0,$$

where $\mu^2 = 1/\epsilon << 1$ and $k = \delta/\epsilon$. Then WKB theory may be used to show that all solutions are bounded if

$$k + \cos t + \cos \omega t > 0 \quad \forall t,$$

that is, if $k > 2$. This result has been verified by numerical integration of eq.(2) for large values of ϵ.