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Averaging using elliptic functions: approximation of limit cycles

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Summary. We apply the method of averaging to first order in the small parameter ε to the autonomous system

$$x'' + \alpha x + \beta x^3 + \varepsilon g(x, x') = 0$$

where we do not consider β as small. This involves perturbing off of Jacobian elliptic functions, rather than off of trigonometric functions as is usually done. The resulting equations involve integrals of elliptic functions which are evaluated using a program written in the computer algebra system MACSYMA. The results are applied to the problem of approximating limit cycles in the above differential equation.

1 Introduction

A limitation of most texts which treat nonlinear vibration problems by perturbation methods is that most problems involve perturbing off of the sine and cosine solutions of simple harmonic oscillators (See [1]—[9]). In this paper we treat a class of problems which involve perturbing off of Jacobian elliptic functions. We consider the differential equation

$$x'' + \alpha x + \beta x^3 + \varepsilon g(x, x') = 0, \qquad \alpha \geqq 0, \quad \beta \geqq 0 \tag{1}$$

in which β is *not* assumed to be a small quantity. We use the method of averaging implemented on MACSYMA to treat this type of problem. We compare results found using elliptic functions with those found using trigonometric functions. In particular we apply our results to the problem of approximating limit cycles in Eq. (1). It is shown that the use of elliptic functions gives better quantitative and in some cases better qualitative results than comparable results obtained by using trigonometric functions.

As an example of the difference between these two approaches, we offer the following problem based on the nonlinear oscillator:

$$x'' + x = \varepsilon \left(-x^3 + \frac{1}{2} x' + \frac{31}{10} x^2 x' - x'^3\right), \qquad \text{with} \quad \varepsilon = \frac{1}{10}. \tag{2}$$

The usual approach (based on trigonometric functions) to studying Eq. (2) involves assuming that the parameter ε is small, and perturbing off of the associated equation (for $\varepsilon = 0$)

$$x'' + x = 0 \tag{3}$$

1*

which has the general solution

$$x = C \cos (t + B). \tag{4}$$

The method of averaging [1]—[9] seeks a solution to Eq. (2) when $\varepsilon \neq 0$ in the form

$$x = C(t) \cos \psi(t). \tag{5}$$

Variation of parameters and averaging over the unperturbed period 2π gives the usual formulas:

$$C' = \frac{\varepsilon}{2\pi} \int_0^{2\pi} G(x, x') \sin \psi \, d\psi \tag{6.1}$$

$$\psi' = 1 + \frac{\varepsilon}{2\pi C} \int_0^{2\pi} G(x, x') \cos \psi \, d\psi, \tag{6.2}$$

in which Eq. (2) has been written in the form

$$x'' + x + \varepsilon G(x, x') = 0. \tag{7}$$

Evaluating Eqs. (6) with $G(x, x') = x^3 - \frac{1}{2} x' - \frac{31}{10} x^2 x' + x'^3$ gives

$$C' = \frac{\varepsilon}{80} C(C^2 + 20) \tag{8.1}$$

$$\psi' = 1 + \frac{3}{8} \varepsilon C^2. \tag{8.2}$$

Nontrivial fixed points of Eq. (8.1) are, in view of (5), periodic motions (limit cycles) of Eq. (2). Since the only fixed point of (8.1) is $C = 0$, the method of averaging predicts that there are no limit cycles for Eq. (2). This prediction is, however, erroneous! See Fig. 1 which shows the results of numerically integrating Eq. (2).

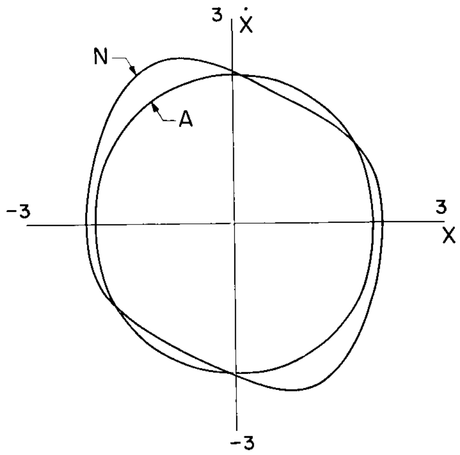

Fig. 1. Limit cycle of Eq. (2) obtained by numerical integration (N). Also shown is the analytic approximation (A) for the limit cycle obtained by using first order averaging utilizing elliptic functions, to be discussed later, see Eq. (29). Note that first order averaging utilizing trigonometric functions fails to predict a limit cycle in this case, cf. Eq. (8.1).

The failure of first order averaging arises from the nature of the limit cycle. In Eq. (2), the limit cycle bifurcates from infinity at $\varepsilon = 0$ and its amplitude scales as $1/\varepsilon$. Therefore, for ε small, the limit cycle exists in that part of the phase space where $x = 0(1/\varepsilon)$. But when x is this large, second order effects (that were neglected before) become important. Averaging to second order in ε can remedy this failure and, in fact, was used to deduce the foregoing conclusion regarding the size of the limit cycle for small ε. Second order averaging involves combining the averaging process with a near-identity transformation of dependent variables. This approach has been treated in [7], where computer algebra (MACSYMA) programs have been presented in order to automate the process. Alternatively, one may stay with first order averaging, but follow the path presented in this paper. We will return to this sample problem later. We also note that results derived from the elliptic function approach hold for all β, even β large.

Although the method of averaging has been treated in numerous references (e.g. [1] to [9]), most treatments deal almost exclusively with perturbations off of the simple harmonic oscillator. A few authors have considered perturbations off of nonlinear systems using elliptic functions. Kuzmak [10] looks for periodic solutions in Eq. (1) using a multiple scale method, where α and β are slowly varying parameters. Garcia-Margallo and Bejarano [11] find limit cycles in a generalized van der Pol oscillator using generalized harmonic balance. Chirikov [12] studies resonance overlap in multiple harmonic excitations of Eq. (1). Davis [13] investigates second order ordinary differential equations using elliptic functions. Cap [14] applies the method of averaging to perturbations of the mathematical pendulum. Pocobelli [15] studies the slowly varying pendulum. Greenspan and Holmes [16] and Guckenheimer and Holmes [1] apply the Melnikov method to perturbations of Eq. (1) where $\alpha < 0$. Nayfeh [5], Kevorkian and Cole [3] and Sanders and Verhulst [8] also treat such problems.

Perturbations of Eq. (1) with $\alpha = 0$ (a purely nonlinear oscillator) have appeared in the literature. Garcia-Margallo and Bejarano [17] employ generalized harmonic balance in order to approximate limit cycles. Yuste and Bejarano [18] use first order averaging as a means to find transitory behavior as motion attracts toward a limit cycle.

In most of these references the authors have reduced the problem to the evaluation of integrals which, through complicated algebraic manipulations, may often be expressed in terms of standard elliptic integrals. By using MACSYMA, we have been able to efficiently evaluate the associated integrals.

We begin with a general treatment of averaging applied to systems of the form (1). For readers unfamiliar with elliptic functions, a brief review is provided in Appendix A. We then apply the method to the problem of approximating limit cycles in Eq. (1).

2 The unperturbed solution

We shall consider unperturbed systems of the form

$$x'' + \alpha x + \beta x^3 = 0. \tag{9}$$

The general solution can be found by assuming the solution in the form

$$x = C cn(At + B, k) = C cn(u, k) = C cn \tag{10}$$

where A and C are positive constants. The $cn(u, k)$ function is a Jacobian elliptic function. It is a periodic function depending on both its argument u and its modulus k. Its period is equal to $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. We abbreviate $cn(u, k)$ by cn where the dependence of cn on its argument u and modulus k are implicitly understood. (Likewise, all elliptic functions in this paper are abbreviated in this manner.) Substituting (10) into Eq. (9) we find

$$[CA^2(2k^2 - 1) + \alpha C]\, cn + [C^3\beta - 2k^2A^2C]\, cn^3 = 0 \tag{11}$$

where we have used the relation

$$\frac{\partial^2 cn(u, k)}{\partial u^2} \equiv cn'' = (2k^2 - 1)\, cn - 2k^2 cn^3. \tag{12}$$

For nontrivial solutions ($C \neq 0$), we find

$$A^2(1 - 2k^2) = \alpha \tag{13.1}$$

$$C^2\beta = 2k^2A^2. \tag{13.2}$$

We then solve Eqs. (13) to find

$$A^2 = \alpha + \beta C^2, \qquad k^2 = \frac{\beta C^2}{2A^2} = \frac{\beta C^2}{2(\alpha + \beta C^2)}. \tag{14}$$

The modulus k and the instantaneous frequency A are known in terms of the amplitude C. Hence, C and the phase angle B are the two undetermined constants which specify the initial conditions.

In applying the averaging method, we choose to use the variables (C, φ) where φ is given by

$$4K(k)\, \varphi = At + B = u. \tag{15}$$

This choice of variables leads to periodic variational equations that can then be averaged. A brief discussion of the (C, φ) variables is given in Appendix B. Thus, the unperturbed solution can be written as

$$x = Ccn(4K\varphi, k) \equiv Ccn \tag{16.1}$$

$$x' = CAcn'(4K\varphi, k) \equiv CAcn' \tag{16.2}$$

$$\begin{gathered} K = K(k), \qquad k = k(C), \qquad A = A(C), \qquad cn' \equiv \frac{\partial cn(u, k)}{\partial u} \\ k^2 = \frac{\beta C^2}{2A^2}, \qquad A^2 = \alpha + \beta C^2 \end{gathered} \tag{16.3}$$

which can be viewed as a generalized van der Pol transformation from (x, x') to (C, φ). In this way, (C, φ) constitute "natural" variables because they take into account the change of period occurring from orbit to orbit in the unperturbed flow.

3 Variation of parameters

In order to obtain a solution to Eq. (1) when $\varepsilon \neq 0$, we vary the parameters (C, φ) so that $C = C(t)$ and $\varphi = \varphi(t)$ in Eqs. (16). Differentiating x in (16.1) and equating the result to (16.2), we obtain

$$\frac{dC}{dt}\left(cn + Ccn'4\varphi K'k' + C\,\frac{\partial cn}{\partial k}\,k'\right) + Ccn'4K\,\frac{d\varphi}{dt} = CAcn' \tag{17}$$

where primes denote differentiation with respect to the argument. (For cn, primes denote differentiation with respect to the argument u and not the modulus k). Differentiating Eq. (16.2), we find

$$x'' = \frac{dC}{dt}\left[(A + A'C)\,cn' + 4CAK'k'\varphi cn'' + CAk'\,\frac{\partial cn'}{\partial k}\right] + 4CAKcn''\,\frac{d\varphi}{dt}. \tag{18}$$

We substitute Eqs. (18) and (16.1) into Eq. (1) and solve for dC/dt and $d\varphi/dt$. Making use of the following identities [19]:

$$cn'^2 = (1 - cn^2)\,(1 - k^2 + k^2cn^2) \tag{19.1}$$

$$cn'\,\frac{\partial cn'}{\partial k} = \frac{1}{2}\,\frac{\partial}{\partial k}\,(cn'^2) = \frac{1}{2}\,\frac{\partial}{\partial k}\,\big((1 - cn^2)\,(1 - k^2 + k^2cn^2)\big) \tag{19.2}$$

we find

$$\frac{dC}{dt} = -\varepsilon\,\frac{1}{A}\,gcn' \tag{20.1}$$

$$\frac{d\varphi}{dt} = \frac{A}{4K} + \varepsilon g\,\frac{1}{4KCA}\left[cn - \frac{(1-2k^2)}{(1-k^2)}\,\big(Zcn' + k^2cn(1 - cn^2)\big)\right] \tag{20.2}$$

$$A = A(C), \qquad k = k(C) \quad \text{both given by (16);} \qquad u = 4K\varphi \tag{20.3}$$

$$K = K(k), \qquad cn \equiv cn(u, k), \qquad cn' \equiv \frac{\partial cn(u, k)}{\partial u}, \qquad Z \equiv Z(u, k). \tag{20.4}$$

The function $Z(u, k)$ denotes the Jacobi Zeta function. In Eq. (20.2), we have used [19]:

$$\frac{\partial cn}{\partial k} = \frac{cn'}{k(1-k^2)}\,[(1 - k^2)\,4K\varphi - E(4K\varphi, k)] - \frac{k}{(1-k^2)}\,cn(1 - cn^2) \tag{21.1}$$

$$Z(4K\varphi, k) = E(4K\varphi, k) - 4\varphi E \tag{21.2}$$

where $E(4K\varphi, k)$ is shorthand notation for $E(\theta, k)$, the incomplete elliptic integral of the second kind (where $\theta = am(4K\varphi, k)$ and $am(u, k)$ is the elliptic amplitude function [19]) and $E = E(k)$ denotes the complete elliptic integral of the second kind. For $\alpha \neq 0$, we also find an auxiliary equation on k:

$$\frac{dk}{dt} = \frac{\alpha\sqrt{\beta}}{\sqrt{2}(\alpha + \beta C^2)^{3/2}}\,\frac{dC}{dt} = -\varepsilon g\,\frac{\sqrt{\beta}}{\sqrt{2}\alpha}\,(1 - 2k^2)^2\,cn', \qquad \alpha \neq 0. \tag{22}$$

Note that Eqs. (20) reduce to the variational equations associated with Eq. (7) and Eqs. (5) for $\beta = 0$ with $\psi = 2\pi\varphi$.

4 The averaging procedure

While Eqs. (20) are valid for any perturbation g, in this section we consider perturbations of the form $g = g(x, x')$, where g is a polynomial in x and x'. We write Eqs. (20) in the form

$$C' = \varepsilon F_1(C, \varphi) \tag{23.1}$$

$$\varphi' = \frac{1}{4K}(\alpha + \beta C^2)^{1/2} + \varepsilon F_2(C, \varphi) = \Omega(C) + \varepsilon F_2(C, \varphi) \tag{23.2}$$

where the F_i, as given by Eqs. (20), are periodic in φ.

We denote the averaged variables by $(\bar{C}, \bar{\varphi})$. Then, the averaged equations become

$$\bar{C}' = \varepsilon \bar{F}_1 + 0(\varepsilon^2) \tag{24.1}$$

$$\bar{\varphi}' = \Omega(\bar{C}) + \varepsilon \bar{F}_2 + 0(\varepsilon^2) \tag{24.2}$$

where $\bar{F}_i$ are the mean values of F_i over one period of the unperturbed system:

$$\bar{F}_i = \frac{1}{T}\int_0^T F_i \, d\varphi = \frac{1}{4\bar{K}}\int_0^{4\bar{K}} F_i(\bar{C}, \bar{u}) \, d\bar{u}, \tag{25}$$

where $\bar{u} = 4\bar{K}\bar{\varphi}$, $\bar{K} = K(\bar{k})$, $\bar{k} = k(\bar{C})$ as given by Eqs. (16).

5 Computer algebra implementation of the averaging scheme

We present a short summary of our implementation of the averaging scheme on the computer algebra system MACSYMA. The perturbation g is composed of a sum of terms of the form

$$x^n x'^m = C^{n+m} A^m cn^n cn'^m \tag{26}$$

each of which can be written as a sum of terms of the form

$$C^{n+m} A^m cn^{n+(m-1)/2} cn', \qquad m \text{ odd} \tag{27.1}$$

$$C^{n+m} A^m cn^{n+m/2}, \qquad m \text{ even} \tag{27.2}$$

using Eq. (19.1). It is therefore sufficient to consider g to be composed of a sum of terms of the form cn^m and $cn^m cn'$. By inspection of Eqs. (20), we can make a list of all combinations of elliptic functions which can possibly occur in the integrands of Eqs. (23), and their mean values. The integrands are listed in Table 1 and their mean values in Table 2.

Table 1. Terms occurring in F_i

Expression	Typical terms
(a) F_1	cn^m, $cn^m cn'$
(b) F_2	cn^m, $cn^m cn'$, Zcn^m, $Zcn^m cn'$

Table 2. Mean values of elliptic functions

Function	Mean Value
(a) cn^m	$\begin{bmatrix} D_m \text{ for } m \text{ even} \\ 0 \text{ for } m \text{ odd} \end{bmatrix}$
(b) $cn^m cn'$	0
(c) Zcn^m	0
(d) $Zcm^m cn'$	$\begin{bmatrix} 0 \text{ for } m \text{ even} \\ -\dfrac{1}{m+1}\left[\left(1-k^2-\dfrac{E}{K}\right)D_{m+1}+k^2D_{m+3}\right], m \text{ odd} \end{bmatrix}$

where

$D_0 = 1,$

$$D_2 = \frac{1}{k^2}\left(\frac{E}{K} - 1 + k^2\right),$$

$$D_m = \frac{1}{(m-1)\,k^2}\,[(m-2)(2k^2-1)\,D_{m-2} + (m-3)(1-k^2)\,D_{m-4}],$$

$m = 4, 6, \ldots$

Armed with Table 2, one could find the averaged equations for a given perturbation $g(x, x')$ by hand. This lengthy calculation, however, is much better suited to MACSYMA. The MACSYMA program which implements the foregoing averaging procedure is listed in Appendix *C*. As an example of its use, we next apply the method to the problem of approximating limit cycles in Eq. (1). We begin by returning to Eq. (2), then we generalize the example.

6 Examples

Eq. (2) revisited

If we write Eq. (2) in the form

$$x'' + x + \frac{1}{10}x^3 = \varepsilon\left(\frac{1}{2}x' + \frac{31}{10}x^2x' - x'^3\right), \qquad \text{with } \varepsilon = \frac{1}{10}$$

we identify $\alpha = 1$, $\beta = \frac{1}{10}$, $g = -\frac{1}{2}x' - \frac{31}{10}x^2x' + x'^3$. Substitution of these values into Eqs. (20) and averaging gives (see sample run of our MACSYMA program in Appendix *C*):

$$\bar{C}' = -\varepsilon\,\frac{P(\bar{C})\,K - Q(\bar{C})\,E}{350\bar{C}K}, \qquad \bar{\varphi}' = \frac{(1+\bar{C}^2/10)^{1/2}}{4K} \tag{28}$$

where

$$P(\bar{C}) = 5\bar{C}^6 + 447\bar{C}^4 + 10175\bar{C}^2 + 64700$$

$$Q(\bar{C}) = 594\bar{C}^4 + 11880\bar{C}^2 + 64700$$

$$\bar{k}^2 = \frac{\bar{C}^2}{2\bar{C}^2 + 20}$$

and where $K = K(\bar{k})$ and $E = E(\bar{k})$ represent complete elliptic integrals of the first and second kinds respectively. Numerical evaluation of the condition $\bar{C}' = 0$ gives the limit cycle amplitude $\bar{C} \simeq 1.9861$. Then Eq. (16.1) gives the following approximation for the limit cycle:

$$x = 1.9861 cn(1.1808t, 0.37608). \tag{29}$$

This approximation offers reasonable agreement with numerical integration of Eq. (2) for $\varepsilon = 1/10$, see Fig. 1. Note that first order averaging off of the simple harmonic oscillator failed to predict a limit cycle for this equation, cf. Eq. (8.1).

Limit cycles in Eq. (1)

We investigate limit cycle solutions in systems of the form

$$x'' + \alpha x + \beta x^3 + \varepsilon g = 0, \qquad \alpha \geqq 0, \quad \beta \geqq 0 \tag{30}$$

in which

$$g = \delta x' + \sum \nu_{ij} x^i x'^j, \qquad \text{where} \quad 2 \leqq i + j \leqq 4.$$

Using Eq. (19.1), Eq. (20.1), and Table 2, we find that the only terms that make nonzero contributions to C' are

$$\delta x', \nu_{21} x^2 x', \nu_{03} x'^3. \tag{31}$$

The condition for a limit cycle is that C' be zero at a particular value of C (but not identically zero), i.e., $\bar{F}_1 = 0$. This condition on the parameters δ, ν_{21}, and ν_{03} will then determine the limit cycle radius (if a limit cycle exists). The other ten terms in g do not influence the existence of a limit cycle (to $O(\varepsilon)$). Therefore, we take a modified perturbation for g:

$$g = \delta x' + \varrho x^2 x' + \eta x'^3. \tag{32}$$

This perturbation includes the example given by Eq. (2). Note also that $\delta = \varrho = \eta = 0$ implies the existence of a family of closed orbits, and not limit cycles.

We find F_1 and $\bar{F}_1$ to be (cf. Eqs. (23), (24)):

$$F_1 = -C[\delta cn'^2 + \varrho C^2 cn^2 cn'^2 + \eta C^2(\alpha + \beta C^2)\, cn'^4] \tag{33.1}$$

$$\bar{F}_1 = -\bar{C}[\delta V_1 + \varrho \bar{C}^2 V_2 + \eta \bar{C}^2(\alpha + \beta \bar{C}^2)\, V_3] \tag{33.2}$$

where

$$V_1 = \text{mean of } cn'^2$$
$$= -\frac{1}{3k^2 K}[K(k^2 - 1) + E(1 - 2k^2)]$$

$$V_2 = \text{mean of } cn^2 cn'^2$$
$$= -\frac{1}{15k^4 K}[K(k^4 - 3k^2 + 2) - 2E(k^4 - k^2 + 1)]$$

$$V_3 = \text{mean of } cn'^4$$
$$= \frac{1}{35k^4 K}[K(8k^6 - 13k^4 + 3k^2 + 2) - 2E(8k^6 - 12k^4 + 2k^2 + 1)].$$

We have dropped the bar notation in V_i for convenience and do so in what follows. The value of k is related to C by Eq. (16.3). The V_i turn out to be positive for valid values of k. Ignoring the trivial case $C = 0$, a limit cycle exists ($\bar{F}_1$ becomes zero) for:

$$\delta V_1 + \varrho C^2 V_2 + \eta C^2(\alpha + \beta C^2) V_3 = 0. \tag{34}$$

Equation (34) is viewed as a relationship between the limit cycle amplitude C and the parameters δ, ϱ, and η.

For the limiting cases of $\alpha = 0$ and of $\beta = 0$, C does not depend on k. Equation (34) can then be solved explicitly for C in terms of the parameters δ, ϱ, and η. In general, however, C does depend on k so that Eq. (34) only implicitly defines C. Equation (34) is solved numerically for k using

$$C^2 = \frac{\alpha}{\beta} \frac{2k^2}{1 - 2k^2}. \tag{35}$$

A bifurcation occurs along the curve that is the intersection of Eq. (34) with

$$\frac{d}{dC} \{\delta V_1 + \varrho C^2 V_2 + \eta C^2(\alpha + \beta C^2) V_3\} = 0. \tag{36}$$

Limit cycles on this curve are degenerate. Upon crossing this bifurcation curve, two limit cycles coalesce at a finite non-zero radius. We continue the discussion by considering the limiting cases.

Case I: results for the linear oscillator ($\beta = 0$)

The values of V_i become indeterminate at $k = 0$. By taking limits we find (cf. Eq. (34)):

$$V_1 = \frac{1}{2}, \qquad V_2 = \frac{1}{8}, \qquad V_3 = \frac{3}{8} \tag{37.1}$$

$$C^2 = -\frac{4\delta}{\varrho + 3\alpha\eta}. \tag{37.2}$$

This agrees with the solution found in [7] by perturbing off of the linear oscillator.

Case II: results for the purely nonlinear oscillator ($\alpha = 0$)

We evaluate V_i and C^2 to be (cf. eqs. (34)):

$$V_1 = \frac{1}{3}, \qquad V_2 \simeq .09139, \qquad V_3 = \frac{1}{7} \tag{38.1}$$

$$C^2 = -\frac{1}{3V_2}\frac{\delta}{\varrho}, \qquad \eta = 0 \tag{38.2}$$

$$C^2 = \frac{-7\sqrt{3}\varrho V_2 \pm \sqrt{7}\sqrt{21\varrho^2 V_2{}^2 - 4\beta\delta\eta}}{2\sqrt{3}\beta\eta}, \qquad \eta \neq 0. \tag{38.3}$$

We continue the discussion of this problem by considering the number of limit cycles which occur for given values of the parameters, i.e., the bifurcation set.

The bifurcation set for $\eta \neq 0$

We consider the bifurcation set of Eq. (34) for $\eta \neq 0$. (The bifurcation set for $\eta = 0$ is easily shown to contain one Hopf bifurcation depending on δ/ϱ and α/β [20].) From Eq. (34), we find

$$\mu_2 = \frac{C^2}{V_1} \left[V_2 \mu_1 + (\alpha + \beta C^2) V_3 \right] \tag{39}$$

where $\mu_1 = \dfrac{\varrho}{\eta}$ and $\mu_2 = -\dfrac{\delta}{\eta}$ are parameters.

Equation (39) defines a family of straight lines in the (μ_1, μ_2) parameter plane with slopes and intercepts parameterized by α, β, and C. Both the slope and the μ_2-intercept have the value zero at $C = 0$, and increase as C increases.

Case I: $\beta = 0$ (The perturbed linear oscillator)

Equation (39) becomes (cf. Eqs. (16.3), (37.1)):

$$\mu_2 = \frac{1}{4} C^2 (3\alpha + \mu_1) \tag{40}$$

with μ_1-intercept at point P ($\mu_1 = -3\alpha$, $\mu_2 = 0$) for all values of C. A graph of Eq. (40) parameterized by C is given in Fig. 2. There is one limit cycle in regions I and II; there are none in regions III and IV. The $\mu_2 = 0$ line is a Hopf bifurcation curve where a limit cycle

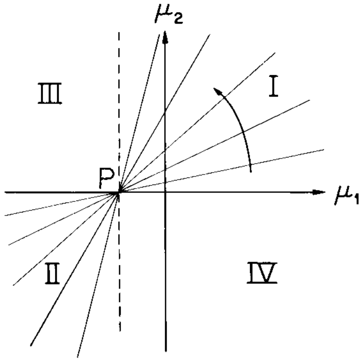

Fig. 2. Limit cycles in Eq. (30) for $\beta = 0$. The parameters μ_1 and μ_2 are defined by $\mu_1 = \varrho/\eta$ and $\mu_2 = -\delta/\eta$, cf. Eq. (40). Along each straight line there exists a limit cycle of fixed amplitude. Thus, in regions I and II there exists 1 limit cycle while in regions III and IV there are no limit cycles. The μ_1 axis corresponds to the limiting case of a limit cycle of zero amplitude (and, hence, a Hopf bifurcation occurs upon crossing the μ_1 axis). The dashed line is $\mu_1 = -3\alpha$ and corresponds to limit cycles of infinite amplitude. The arrow shows the direction of increasing limit cycle amplitude.

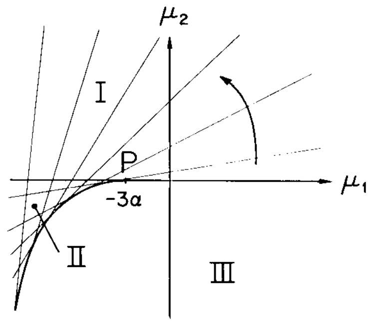

Fig. 3. Limit cycles in Eq. (30) for $\beta \neq 0$. The parameters μ_1 and μ_2 are defined by $\mu_1 = \varrho/\eta$ and $\mu_2 = -\delta/\eta$, cf. Eq. (39). Along each straight line there exists a limit cycle of fixed amplitude. Thus, in region I there exists 1 limit cycle; in region II there exist 2 limit cycles; and in region III there are no limit cycles. The μ_1 axis corresponds to the limiting case of a limit cycle of zero amplitude (and, hence, a Hopf bifurcation occurs upon crossing the μ_1 axis). Along the curve separating region II from III two limit cycles coalesce. The arrow shows the direction of increasing limit cycle amplitude.

is born at the origin. On the bifurcation line $\mu_1 = -3\alpha$, a limit cycle of infinite amplitude is predicted. Point P is a highly singular point: near P, the limit cycle amplitude is very sensitive to small changes in μ_1 and μ_2.

Case II: $\beta \neq 0$ (The perturbed nonlinear oscillator)

The μ_1 intercept moves out from $\mu_1 = -3\alpha$ at $C = 0$ towards infinity as $C \to \infty$. With this information, we plot Eq. (39), parameterized by C, in the (μ_1, μ_2) plane (see Fig. 3). One limit cycle exists in region I, two in region II (where each point lies on exactly two intersecting lines), and none in region III. A degenerate limit cycle exists on the bifurcation curve between II and III. The μ_1 axis is a Hopf bifurcation curve where a limit cycle is born at the origin. Point P ($\mu_1 = -3\alpha$, $\mu_2 = 0$) is again a singular point where a degenerate limit cycle of zero amplitude exists. Near P, the sensitivity of the amplitude on μ_1 and μ_2 depends on the smallness of β.

The predictions of Fig. 3 are in agreement with the results of numerical integration of the original differential equation (30).

Comparing Case I with Case II

A comparison of the linear analysis ($\beta = 0$, Fig. 2) with the nonlinear analysis ($\beta \neq 0$, Fig. 3) shows qualitatively different results. In both analyses, a perturbation term of the form $\varepsilon \nu_{30} x^3$ does not contribute to determining the existence of a limit cycle. Yet for β small, *the nonlinear analysis does not reduce to the linear one.* The linear analysis fails to predict one limit cycle in region IV and two limit cycles in part of region II of Fig. 2 for β small.

Numerical simulations confirm the nonlinear analysis. Equation (2) provided an example with the following parameter values:

$$\alpha = 1, \qquad \beta = \varepsilon = 0.1, \qquad \delta = -0.5, \qquad \varrho = -3.1, \qquad \eta = 1 \tag{41}$$

in which the system belongs to region III of Fig. 2 and I of Fig. 3. As we saw before, the analysis based on elliptic functions agreed with numerical integration, while the usual trigonometric approach failed to predict a limit cycle.

Another example is afforded by the parameter values:

$$\alpha = 1, \qquad \beta = 2\varepsilon = 0.1, \qquad \delta = 1, \qquad \varrho = -4.6, \qquad \eta = 1 \tag{42}$$

in which the system belongs to region II of Figs. 2 and 3. A numerical simulation finds two limit cycles with amplitudes 1.93 and 2.93. Using Eq. (39), the predicted values are 1.97 and 2.59, which compare well with the numerical integration values. The first order linear prediction Eq. (37.2) predicts only one limit cycle with amplitude 1.58.

7 Conclusions

With the advent of computer algebra, perturbation analyses using elliptic functions can now be done almost as easily as those using trigonometric functions. We have shown that perturbing off of elliptic functions will generally provide better quantitative and in some cases better qualitative results than a comparable perturbation off of trigonometric functions for systems containing an x^3 term. In some problems, averaging off of elliptic functions (which contain an amplitude-frequency dependence that trigonometric functions lack) provides results at first order which can only be attained by averaging off of trigonometric functions to second order. In the case of limit cycles in Eq. (30), first order trigonometric averaging gives qualitatively incorrect predictions if $\beta \neq 0$ and $\mu_1 < -3\alpha$, cf. Figs. 2 and 3.

Related work in progress by the authors includes the extension of the averaging method off of elliptic functions to include terms of $0(\varepsilon^2)$. This involves computing a near-identity transformation and is a generalization of second order averaging off of trigonometric functions (see [7].) Additional applications of the MACSYMA program have been made to the forced Duffing equation and to systems of the form of Eq. (9) in which α and β are slowly varying functions of time. In particular, extensions of this work to problems in which α and β are not necessarily positive are in progress (see [20]).

Appendix A: Jacobian elliptic functions

Jacobian elliptic functions involve a collection of identities which are similar to those for trigonometric functions but are more complicated algebraically. The use of computer algebra makes manipulation of these identities easier, permitting investigations to proceed on problems which were previously avoided because of the quantities of algebra involved. All formulas and conventions concerning Jacobian elliptic functions in this paper are taken from [19].

We now offer a brief comparison of elliptic functions with the more familiar trigonometric functions. Corresponding to sin (u) and cos (u) are three fundamental elliptic functions sn (u, k), cn (u, k), and dn (u, k). Each of the elliptic functions depends on the modulus k as well as the argument u. These reduce to sin (u), cos (u), and 1 respectively, when $k = 0$. The sn and sin functions share common properties as do cn and cos. These are summarized in Table 3. The dn function has no trigonometric counterpart. Note that the elliptic functions sn and cn be thought of as generalizations of sin and cos where their period depends on the modulus k.

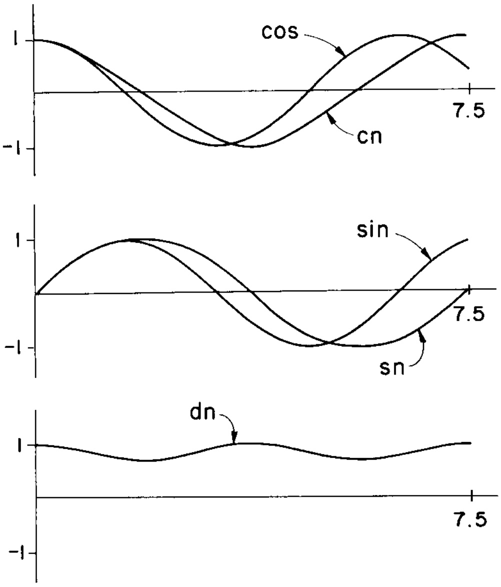

Fig. 4. Comparison of elliptic functions for $k = \sqrt{1/2}$ with trigonometric functions. The period of the elliptic functions is $4K(\sqrt{1/2}) \simeq 7.416$. See Table 3.

Table 3. Properties of Jacobian elliptic functions

Function f					
Property	$sn(u, k)$	$\sin(u)$	$cn(u, k)$	$\cos(u)$	$dn(u, k)$
Max. value	1	1	1	1	1
Min. value	-1	-1	-1	-1	$(1-k^2)^{1/2}$
Period	$4K(k)$	2π	$4K(k)$	2π	$2K(k)$
Odd/Even	Odd	Odd	Even	Even	Even
df/du	$cn\, dn$	cos	$-sn\, dn$	$-\sin$	$-k^2\, sn\, cn$
$f \mid k = 0$	sin	sin	cos	cos	1

$K(k)$ = complete elliptic integral of the first kind

$K(0) = \pi/2, \quad K(1) = +\infty$

The argument u is identified as the incomplete elliptic integral of the first kind which is usually denoted $F(\theta, k)$. This identification shows that u also depends on k. The value of k normally ranges from 0 to 1. The sn, cn, and dn functions are shown in Fig. 4 for $k = \sqrt{1/2}$.

The elliptic functions also satisfy the following identities which correspond to $\sin^2 + \cos^2 = 1$:

$$sn^2 + cn^2 = 1 \tag{A.1}$$

$$k^2 sn^2 + dn^2 = 1 \tag{A.2}$$

$$1 - k^2 + k^2 cn^2 = dn^2. \tag{A.3}$$

In addition to the sn, cn, and dn functions, there are three other frequently encountered elliptic functions. First, there is the amplitude function $am(u, k)$ ($= \theta$) which is the inverse of $F(\theta, k)$ ($= u$). This function maps the elliptic argument u onto a trigonometric argument θ so that the period $4K(k)$ in u equals the period 2π in θ.

Second, there is $E(\theta, k)$, the incomplete elliptic integral of the second kind. It is often written in abbreviated notation as $E(u)$ since θ depends on u (via the am function) and the dependence on k is understood. Both $E(u)$ and u are not periodic in u. The complete elliptic integral of the second kind is denoted $E(k)$.

Finally, a linear combination of $E(u)$ and u is periodic, and called the Jacobi Zeta function $Z(\theta, k)$:

$$Z(\theta, k) = E(\theta, k) - \frac{E(k)}{K(k)} F(\theta, k). \tag{A.4}$$

This function is periodic with period $2K(k)$ and has zero mean. It is often written in abbreviated notation as $Z(u, k)$.

Appendix B: The averaging variables

The averaging method is composed of two parts: (i) deriving perturbation equations which are periodic in some angle variable through variation of parameters, and (ii) averaging these equations over one period in the angle where the amplitude variable is held "fixed". In contrast to perturbations of the linear oscillator, the choice of the angle variable is crucial to the method.

First, note that since Eq. (9) is Hamiltonian, it possesses the first integral

$$H = \frac{1}{2} x'^2 + \frac{1}{2} \alpha x^2 + \frac{1}{4} \beta x^4 \tag{B.1}$$

which provides another method for solving Eq. (9). We define action-angle variables (J, φ) for this Hamiltonian [1], [21] in order to provide more "natural" variables to be used in setting up the averaging scheme. After some lengthy calculations, we find that

$$J = \oint x' \, dx = \frac{4}{3} A \frac{C^2}{k^2} [(2k^2 - 1) E(k) + (1 - k^2) K(k)] \tag{B.2}$$

$$4K(k) \varphi = At + B = u \tag{B.3}$$

$A = A(C)$ and $k = k(C)$ given by Eq. (16.3).

We consider J to be a function only of C (and the parameters α and β). Since Eq. (B.2) is not explicitly invertible, we choose C as the variable to be used in the averaging scheme.

The variable φ is preferred to u and B in deriving the perturbation equations because while the perturbation equations are periodic in φ, they are not in u or B. This is seen as follows. Although each orbit in phase space is orbitally stable [2], [4], [9], it is Lyapunov unstable. This is because the frequency of an orbit depends on its amplitude, and motions starting close together but on two different orbits eventually become far apart (i.e., out of phase), even though their orbits are close. This "phase shear" instability is reflected in the perturbation equations for u and B.

In deriving the perturbation equations based on using (C, B), we would obtain equations of the form

$$C' = \varepsilon f_1(C, At + B), \qquad B' = -A't + \varepsilon f_2(C, At + B) \tag{B.4}$$

in which f_2 turns out not to be periodic in t. Thus Eqs. (B.4) are unsuitable for averaging. The orbital stability of the solutions is reflected in the equation for C. The Lyapunov (phase) instability is reflected in the equation for B.

Similarly, choosing (C, u) as primitive variables, $u = At + B$, gives

$$C' = \varepsilon f_1(C, u), \qquad u' = A + \varepsilon f_3(C, u) \tag{B.5}$$

in which f_3 is not periodic in u, so that Eqs. (B.5) are again unsuitable for averaging.

However, setting $u = 4K\big(k(C)\big) \varphi$, cf. (B.3), gives

$$C' = \varepsilon F_1(C, 4K\varphi), \qquad \varphi' = \frac{A}{4K} + \varepsilon F_2(C, 4K\varphi) \tag{B.6}$$

in which both F_1 and F_2 are found to be periodic in φ and hence in the correct form for averaging. Thus, we use (C, φ) as the averaging variables.

Appendix C: MACSYMA Computer program listing

```
/* ROUTINE TO PERTURB OFF X'' + AL X + BE X^3 + E G(X,X') = 0 */
AVERAGE():=BLOCK([X,Y,XX,YY,EC,KC,AL,BE,G,F,FX2,FZ2,FI,FBAR,HI,D,
CFLOW,PFLOW],
```

```
PRINT("AVERAGING OF X'' + AL X + BE X^3 + EPS G X,X',EPS*T"),
PRINT(" "),AL:READ("ENTER AL:"),
PRINT(" "),BE:READ("ENTER BE:"),
PRINT(" "),PRINT("ENTER G(X,X') USING Y=X':"),
G:READ( ),
PRINT(" "),PRINT("THE SOLUTION TO THE UNPERTURBED SYSTEM IS"),
PRINT("X = C CN(4*KC(C)*PHI,K)"),
PRINT("X' = C SQRT(AL + BE C^2) CN'(4*KC(C)*PHI,K)"),
PRINT("WHERE 0 <= K^2 = BE C^2/2/(AL +BE C^2) <= 1/2"),
PRINT("KC = COMPLETE ELLIPTIC INTEGRAL OF FIRST KIND"),
PRINT("AND 4*KC(K)*PHI = SQRT(AL + BE C^2)*T+B"),PRINT(" "),
PRINT("SEEK PERTURBED SOLUTION OF SAME FROM WHERE (C,PHI)"),
PRINT("BECOME FUNCTIONS OF TIME"),
PRINT(" "),

/* X = C CN(4*KC*PHI)       */
/* Y = X' = C SQRT(AL + BE C^2) CN'(4*KC*PHI)       */

/* SYMBOLS */

/* XX = CN FUNCTION                                              */
/* YY = CN' FUNCTION (DERIVATIVE OF CN W.R.T. ARGUMENT) */
/* ZZ = ZETA FUNCTION                                            */
/* KC,EC = COMPLETE ELLIPTIC INTEGRALS OF 1ST,2ND KINDS */
/* K = MODULUS                                                    */

KILL(K),

/* FOR SPECIAL CASES, K IS A NUMBER */

IF AL = 0 THEN K:SQRT(1/2),
IF BE = 0 THEN (K:0,KC:EC:%PI/2),

/* REDUC ROUTINE TO REDUCE EXPRESSIONS TO FORMS: CN^M AND
CN^M CN' */

REDUC(EXPR):=BLOCK([EVEN,ODD,VAL],
      EVEN:EXPAND((EXPR+EV(EXPR,YY=-YY))/2),
      ODD:EXPAND((EXPR-EVEN)/YY),
      ODD:YY*EXPAND(EV(ODD,YY=SQRT((1-XX^2)*(1-K^2+K^2
*XX^2)))),
      EVEN:EXPAND(EV(EVEN,YY=SQRT((1-XX^2)*(1-K^2+K^2*XX^2)))),
      VAL:EVEN+ODD
),

/* AVERAGING PROCEDURE */

G:EV(G,X=C*XX,Y=C*SQRT(AL+BE*C^2)*YY),

F[1]:-1/SQRT(AL+BE*C^2)*REDUC(G*YY),
F[2]:1/C/4/KC/SQRT(AL+BE*C^2)
        *REDUC(G*(XX-(1-2*K^2)/(1-K^2)*(ZZ*YY+K^2*XX*(1-XX^2)))),

IF K = 0 THEN F[2]:EV(F[2],ZZ=0),

F[1]:EV(F[1],YY=0),             /* CN^M CN' TERMS HAVE NO MEAN */
```

```
  FZ2:RATCOEF(F[2],ZZ),           /* PICK OFF Z TERMS IN F[2] */
  FX2:EXPAND(F[2]-FZ2*ZZ),        /* PICK OFF X TERMS IN F[2] */
  FZ2:EXPAND(EV(FZ2-EV(FZ2,YY=0),YY=1)),
                                  /* Z CN^M TERMS HAVE NO MEAN */
  FX2:EV(FX2,YY=0),               /* CN^M CN' TERMS HAVE NO MEAN */

  /* MEAN VALUE ROUTINE */

  D[0]:1,
  D[1]:0,
  D[2]:1/KBAR^2*(EC/KC-1+KBAR^2),
  D[3]:0,

  D[II]:=RATSIMP(1/(II-1)/KBAR^2*((II-2)*(2*KBAR^2-1)*D[II-2]
                 +(II-3)*(1-KBAR^2)*D[II-4])),

  IF K = 0 THEN (D[2]:1/2,D[II]:=RATSIMP((II-1)/II*D[II-2])),
  IF K = SQRT(1/2) THEN KBAR:SQRT(1/2),

  /* FIND MEAN USING TABLE 2 */

  HI:MAX(HIPOW(F[1],XX),HIPOW(FX2,XX),HIPOW(FZ2,XX)),
  FOR II:1 THRU 2 DO FBAR[II]:0,
  FOR II:0 THRU HI DO (
      FBAR[1]:FBAR[1]+RATCOEF(F[1],XX,II)*D[II],
      FBAR[2]:FBAR[2]+RATCOEF(FX2,XX,II)*D[II]
            -RATCOEF(FZ2,XX,II)/(II+1)*((1-KBAR^2-EC/KC)*D[II+1]
                                       +KBAR^2*D[II+3])
  ),

  /* CHANGE RESULTS TO PRINTABLE FORM */

  FOR II:1 THRU 2 DO FBAR[II]:EV(FBAR[II],ABS(C)=CBAR,C=CBAR,
  K=KBAR),

  /* PRINT AVERAGED EQS */

  CFLOW:EPS*FACTOR(FBAR[1]),
  PFLOW:1/4/KC*EV(SQRT(AL+BE*CBAR^2),ABS(CBAR)=CBAR)+EPS
  *FACTOR(FBAR[2]),

  DERIVABBREV:TRUE,KILL(KBAR),
  VAL:[DIFF(CBAR(T),T)=CFLOW,DIFF(PHIBAR(T),T)=PFLOW,
      KBAR^2 = BE*CBAR^2/2/(AL+BE*CBAR^2)],

  PRINT("THE AVERAGED EQUATIONS ARE"),PRINT(" "),
  PRINT(VAL),PRINT(" ")
)$
```

Here is a sample run based on the example discussed in the text, Eq. (2):

```
(c6) AVERAGE()$

PERTURBATION OF X'' + AL X + BE X^3 + EPS G(X,X',EPS*T) = 0 BY
AVERAGING

ENTER AL:
1;
```

ENTER BE:

1/10;

ENTER G(X,X') USING Y=X':

−Y/2−31*X^2*Y/10+Y^3;

THE SOLUTION TO THE UNPERTURBED SYSTEM IS
X = C CN(4*KC(C)*PHI,K) , X' = C SQRT(AL + BE C^2) CN'(4*KC(C)*PHI,K)
WHERE 0 <= K^2 = BE C^2/2/(AL +BE C^2) <= 1/2
KC = COMPLETE ELLIPTIC INTEGRAL OF FIRST KIND
AND 4*KC(K)*PHI = SQRT(AL + BE C^2)*T+B

SEEK PERTURBED SOLUTION OF SAME FORM WHERE (C,PHI) BECOME FUNCTIONS OF TIME
THE AVERAGED EQUATIONS ARE

$$\begin{aligned}
&[\mathrm{cbar}(t)_t = -\,\mathrm{cbar\ eps}\,(24\,\mathrm{cbar}^4\,\mathrm{kbar}^6\,\mathrm{kc} + 240\,\mathrm{cbar}^2\,\mathrm{kbar}^6\,\mathrm{kc}\\
&- 39\,\mathrm{cbar}^4\,\mathrm{kbar}^4\,\mathrm{kc} - 173\,\mathrm{cbar}^2\,\mathrm{kbar}^4\,\mathrm{kc} + 175\,\mathrm{kbar}^4\,\mathrm{kc} + 9\,\mathrm{cbar}^4\,\mathrm{kbar}^2\,\mathrm{kc}\\
&- 561\,\mathrm{cbar}^2\,\mathrm{kbar}^2\,\mathrm{kc} - 175\,\mathrm{kbar}^2\,\mathrm{kc} + 6\,\mathrm{cbar}^4\,\mathrm{kc} + 494\,\mathrm{cbar}^2\,\mathrm{kc}\\
&- 48\,\mathrm{cbar}^4\,\mathrm{ec}\,\mathrm{kbar}^6 - 480\,\mathrm{cbar}^2\,\mathrm{ec}\,\mathrm{kbar}^6 + 72\,\mathrm{cbar}^4\,\mathrm{ec}\,\mathrm{kbar}^4\\
&+ 286\,\mathrm{cbar}^2\,\mathrm{ec}\,\mathrm{kbar}^4 - 350\,\mathrm{ec}\,\mathrm{kbar}^4 - 12\,\mathrm{cbar}^4\,\mathrm{ec}\,\mathrm{kbar}^2 + 314\,\mathrm{cbar}^2\,\mathrm{ec}\,\mathrm{kbar}^2\\
&+ 175\,\mathrm{ec}\,\mathrm{kbar}^2 - 6\,\mathrm{cbar}^4\,\mathrm{ec} - 494\,\mathrm{cbar}^2\,\mathrm{ec})/(1050\,\mathrm{kbar}^4\,\mathrm{kc}),\\
&\mathrm{phibar}(t)_t = \frac{\mathrm{sqrt}\left(\frac{\mathrm{cbar}^2}{10}+1\right)}{4\,\mathrm{kc}},\quad \mathrm{kbar}^2 = \frac{\mathrm{cbar}^2}{20\left(\frac{\mathrm{cbar}^2}{10}+1\right)}\Bigg]
\end{aligned}$$

(VAX 8530 Time = 157 sec.)

The results of the program give the averaged equations in terms of both $\bar{C}$ (called cbar) and $\bar{k}$ (called kbar). The results are stored in the variable VAL: VAL[1] contains the $\bar{C}'$ equation, VAL[2] contains the $\bar{\varphi}'$ equation and VAL[3] contains the expression for $\bar{k}^2$ in terms of $\bar{C}^2$. The following command substitutes $\bar{k}$ in terms of $\bar{C}$, giving Eq. (39) of the text:

(c7) FACTOR(EV(VAL[1],KBAR=SQRT(RHS(VAL[3]))));

$$\begin{aligned}
(\mathrm{d}7)\ \mathrm{cbar}(t)_t = -\mathrm{eps}\,(&5\,\mathrm{cbar}^6\,\mathrm{kc} + 447\,\mathrm{cbar}^4\,\mathrm{kc} + 10175\,\mathrm{cbar}^2\,\mathrm{kc} + 64700\,\mathrm{kc}\\
&- 594\,\mathrm{cbar}^4\,\mathrm{ec} - 11880\,\mathrm{cbar}^2\,\mathrm{ec} - 64700\,\mathrm{ec})/(350\,\mathrm{cbar}\,\mathrm{kc})
\end{aligned}$$

(VAX 8530 Time = 3 sec.)

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