Nonlinear Vibrations of Two-Degree-of-Freedom Systems With Repeated Linearized Natural Frequencies

R. RAND[1] and R. VITO[2]

EXPLICIT approximate expressions for normal modal curves in nonlinear two-degree-of-freedom systems have been obtained by Rosenberg and Kuo [1][3] and by Rand [2, 3]. This paper extends these results to include a special case previously untreated.

Consider a holonomic, scleronomous conservative system with generalized coordinates x, y. Consider a class of systems for which the potential energy V is of the form

$$V = V_1 + V_2 \tag{1}$$

$$V_1 = \frac{\omega^2}{2}(x^2 + y^2) \tag{2}$$

$$V_2 = \alpha x^4 + \beta x^3 y + \gamma x^2 y^2 + \sigma x y^3 + \tau y^4 \tag{3}$$

where $\omega, \alpha, \beta, \gamma, \sigma, \tau$ are constants such that V is positive-definite and for which the kinetic energy T is of the form

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{4}$$

where dots represent differentiation with respect to time t.

Then the equations of motion become

$$\ddot{x} = -\frac{\partial V}{\partial x} \tag{5}$$

$$\ddot{y} = -\frac{\partial V}{\partial y} \tag{6}$$

If these equations are linearized in the neighborhood of the origin $x = y = 0$; that is, if V_2 is neglected in equation (1), then both linearized natural frequencies are repeated and are equal to ω. The treatment in references [2, 3] fails for this case; various denominators vanish whenever the linearized natural frequencies are repeated. This problem corresponds to the resonance phenomena of small divisors in celestial mechanics [4].

Note that V_2 represents the most general potential corresponding to forces cubic in the generalized coordinates.

Theorem. If C is a real root of the algebraic equation

$$\sigma C^4 + 2(\gamma - 2\tau)C^3 + 3(\beta - \sigma)C^2 + 2(2\alpha - \gamma)C - \beta = 0 \tag{7}$$

then the system represented by equations (1)–(6) possesses a solution of the form

$$y = Cx \tag{8}$$

Proof. A solution of the form (8) is called by Rosenberg [5] a similar normal mode. A necessary and sufficient condition on V for such a solution is

$$C\frac{\partial V}{\partial x}(x, Cx) = \frac{\partial V}{\partial y}(x, Cx) \tag{9}$$

hence equations (5) and (6) become identical. Substituting equations (1)–(3) into (9) yields condition (7), *QED*.

Substituting equation (8) into equation (5) provides a single nonlinear ordinary differential equation for $x(t)$ for which approximate methods of solution are well known [6].

Since V_1 dominates V_2 in some sufficiently small neighborhood of the origin $x = y = 0$, any values of $\alpha, \beta, \gamma, \sigma, \tau$ will leave V positive-definite in such a neighborhood.

By application of Descartes' Rule of Signs the nature of the roots of equation (7) may be found. If β and σ are of the same sign then the roots are either

(*i*) 3 positive, 1 negative.
(*ii*) 1 positive, 3 negative.
(*iii*) 1 positive, 1 negative, 2 complex.

If β and σ are of opposite sign then the roots are either

(*i*) 2 positive, 2 negative.
(*ii*) 2 positive, 2 complex.
(*iii*) 2 negative, 2 complex.
(*iv*) 4 complex.

If β vanishes then $C = 0$ is a root and the other roots are either

(*i*) 2 positive, 1 negative.
(*ii*) 1 positive, 2 negative.
(*iii*) 1 positive, 2 complex.
(*iv*) 1 negative, 2 complex.

If σ vanishes then $C = \infty$ is a root, i.e., $x = 0$ is a similar normal mode. If both β and σ vanish, then $C = 0, \infty$ are roots, and the other roots follow from

$$C^2 = \frac{2\alpha - \gamma}{2\tau - \gamma} \tag{10}$$

Note that if β and σ are of the same sign then there is at least one positive and one negative root. Further, all the roots cannot be real and positive, nor can they all be real and negative.

As an example, consider the system shown in Fig. 1. Two particles of unit mass are constrained to a straight line and restrained by three springs as shown. Let F be the tension in a spring and let δ be its elongation beyond its unstretched length. Then it is assumed that for the identical anchor springs

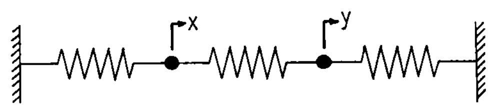

Fig. 1 A two-degree-of-freedom nonlinear system with repeated linearized natural frequencies

[1] Assistant Professor, Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, N. Y.

[2] Graduate Student, Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, N. Y.

[3] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, March 1, 1971.

$$F = \delta + K_1\delta^3 \quad (11)$$

while for the center spring

$$F = K_2\delta^3 \quad (12)$$

where K_1 and K_2 are arbitrary positive constants. If x and y, the displacements from equilibrium, are taken as generalized coordinates, the potential energy V becomes

$$V = [\tfrac{1}{2}x^2 + \tfrac{1}{4}K_1x^4] + [\tfrac{1}{4}K_2(x - y)^4] + [\tfrac{1}{2}y^2 + \tfrac{1}{4}K_1y^4] \quad (13)$$

Comparison with equations (2) and (3) reveals

$$\begin{aligned} \omega &= 1 \\ \alpha = \tau &= (K_1 + K_2)/4 \\ \beta = \sigma &= -K_2 \\ \gamma &= 3K_2/2 \end{aligned} \quad (14)$$

Equation (7) becomes

$$C^4 + (K - 2)C^3 - (K - 2)C - 1 = 0 \quad (15)$$

where $K = K_1/K_2$ can take any value from zero to infinity. Solving equation (15), the four roots are found to be

$$C_1 = 1, \quad C_2 = -1, \quad 2C_{3,4} = 2 - K \pm [K(K - 4)]^{1/2} \quad (16)$$

For $0 \leq K < 4$, C_3 and C_4 are imaginary and only two solutions of the form (8) exist. However, for $K > 4$, C_3 and C_4 are both negative and there are four distinct similar normal modes.

References

1 Rosenberg, R. M., and Kuo, J. K., "Nonsimilar Normal Mode Vibrations of Nonlinear Systems Having Two Degrees of Freedom," JOURNAL OF APPLIED MECHANICS, Vol. 31, No. 2, TRANS. ASME, Vol. 86, Series E, June 1964, pp. 283–290.

2 Rand, R. H., "Nonlinear Normal Modes in Two-Degree-of-Freedom Systems," JOURNAL OF APPLIED MECHANICS, Vol. 38, No. 2, TRANS. ASME, Vol. 93, Series E, June 1971, p. 561.

3 Rand, R. H., "A Higher-Order Approximation for Nonlinear Normal Modes in Two-Degree-of-Freedom Systems," to be published in the *International Journal of Nonlinear Mechanics*.

4 Szebehely, V., *Theory of Orbits*, Academic Press, 1967, p. 386.

5 Rosenberg, R. M., "On Nonlinear Vibrations of Systems With Many Degrees of Freedom," *Advances in Applied Mechanics*, Academic Press, 1966.

6 Minorsky, N., *Nonlinear Oscillations*, Van Nostrand, 1962.