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COMPUTER ALGEBRA AND THE ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES

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Perturbation solutions of differential equations typically involve power series expansions in a small parameter ϵ. Computer algebra has proved a useful tool for the treatment of perturbation problems due to the large quantities of algebra involved [1,2]. As an example, we treat a problem which was previously handled both by numerical integration and by hand-derived perturbation analysis. It is shown that the use of computer algebra yields a vast increase in the number of terms obtained in the perturbation expansion, and raises new questions concerning the convergence of the derived series.

The elliptic restricted problem of three bodies describes the planar motion of a particle (the third body) under the gravitational attraction of two masses (the primaries) which move about their common center of mass in elliptical orbits [3,4,5]. We shall be interested in the stability of the L_4 equilibrium, located at the vertex of an equilateral triangle which is positioned so that the primaries lie at the other two vertices. This problem is governed by the following d.e.'s:

$$(1.1) \qquad x'' - 2\,y' - g\,h_2\,x = 0$$

$$(1.2) \qquad y'' + 2\,x' - g\,h_1\,y = 0$$

where $g = 1/(1 + e \cos f)$ and

$$h_{1,2} = \frac{3}{2}\left[1 \pm [1 - 3\mu(1-\mu)]^{1/2}\right]$$

in which primes represent differentiation with respect to the true anomaly f (the independent variable). Eqs.(1) involve two parameters μ (mass ratio of the smaller primary to the sum of the primaries) and e (eccentricity of the orbit of the primaries). For given μ,e, the equilibrium L_4 is stable if all solutions to (1) are bounded, and unstable if an unbounded solution exists.

Numerous papers have been written since the mid-1960's concerning this problem. We mention only a few. Danby [3] used numerical integration and Floquet theory to determine regions of stability in the μ-e plane, see Fig.1. Alfriend and Rand [4], and later Nayfeh and Kamel [5], used perturbation theory to obtain approximate closed form expressions for the transition curves bounding the stable regions in Fig.1. In this work we shall use computer algebra and an efficient perturbation scheme to extend these previous results.

The transition curves intersect the μ-axis at the points $\mu_a = (1/2)[1-\sqrt{23/27}] = 0.03852\ldots$ and $\mu_b = (1/2)[1-\sqrt{24/27}] = 0.02859\ldots$. We shall describe our procedure for the curve through μ_a. The other two curves through μ_b are found similarly. We expand

$$(2.1) \qquad \mu = \mu_a + \mu_1\,e + \mu_2\,e^2 + \cdots$$

$$(2.2) \qquad x = \cos\sigma f + x_1\,e + x_2\,e^2 + \cdots$$

$$(2.3) \qquad y = -\alpha \sin\sigma f + y_1\,e + y_2\,e^2 + \cdots$$

where $\sigma = 1/\sqrt{2}$ and $\alpha = \sqrt{2} - 1$. Following the treatment of Mathieu's equation in [1], pp.112-114, we set

$$(3.1) \qquad x_n = \sum_{j=-n}^{n} A_{nj} \cos(j+\sigma)f$$

$$(3.2) \qquad y_n = \sum_{j=-n}^{n} B_{nj} \sin(j+\sigma)f$$

Substitution of (2) and (3) into (1), equating to zero coefficients of e^n, using trig identitites to simplify products of sines and cosines, and equating coefficients of $\cos(j+\sigma)f$ and $\sin(j+\sigma)f$ to zero gives a recursive system of linear algebraic equations on the A_{nj}, B_{nj} and μ_n coefficients. Solving the resulting equations on MACSYMA (which uses exact rational arithmetic), we obtained an expression for the curve through μ_a to $O(e^{50})$:

$$(4) \qquad \mu = \mu_a + \sqrt{3/23}\left[\frac{2}{9}e^2 - \frac{2305}{3312}e^4 + \frac{4225723}{1218816}e^6 - \frac{302930227585}{12558680064}e^8 + \frac{289720124239853}{1540531421184}e^{10} + \cdots +\right.$$

1784942618981801354708877788405583336316516202
9
1254619002727682872585343444990772843284328212
1
2012265847024631767966020212117405056905 /
1177477458972221772971990542686939344684648238
3
3866376254049722495867392652896585232013599541
9
176236474725040128 e^{50}]

in which all odd powers of e are absent. In a similar fashion we obtained an expression for the curves through μ_b to $O(e^{45})$:

$$(5) \qquad \mu = \mu_b + \sqrt{1/2} \left[\frac{49}{2304} e^2 - \frac{114275}{7077888} e^4 - \frac{75233555}{10871635968} e^6 - \cdots \right]$$

$$\pm \sqrt{3/22} \left[\frac{11}{72} e - \frac{751}{12288} e^3 - \frac{650461}{46137344} e^5 - \cdots - \right.$$

$$\left. 1857467163383270136926113982227345782659502604943413058368215797950124233685104479241 3 / 37408189466397092126303160224013528354823539771 80171046649256833995639739702858115501260 8 \; e^{45} \right]$$

Eq.(4) was given to $O(e^2)$ in [4] while eq.(5) was given to $O(e^4)$ in [5], although the e^4 coefficient was incorrectly given in [5].

Fig.1 displays the graphs of eqs.(4),(5). The two transition curves (5) through μ_b cannot be distinguished from the results of numerical integration. However, the curve through μ_a, eq.(4), differs considerably from the result obtained by numerical integration. Fig.2 shows the partial sums of eq.(4).

What can account for the disagreement between eq.(4) and the numerical result, especially in view of the excellent agreement exhibited by eq.(5) ? Firstly, the coefficients of eq.(4) may be incorrect. We have checked these results in numerous ways, but we would invite an independent check by other investigators. Secondly, the convergence of the series (4) may be questioned. Using estimates based on the ratio test, on Pade approximants and on reversion of the series, we have found that the radius of convergence of the power series (4) is about $e \cong 0.3$. Moreover, even for values of e for which (4) converges, it need not necessarily converge to the correct transition curve. This could happen, e.g., if the correct expression included terms with zero taylor series, e.g., $\exp(-1/e)$. We are frankly puzzled at the behavior of eq.(4), and we invite comments by interested readers.

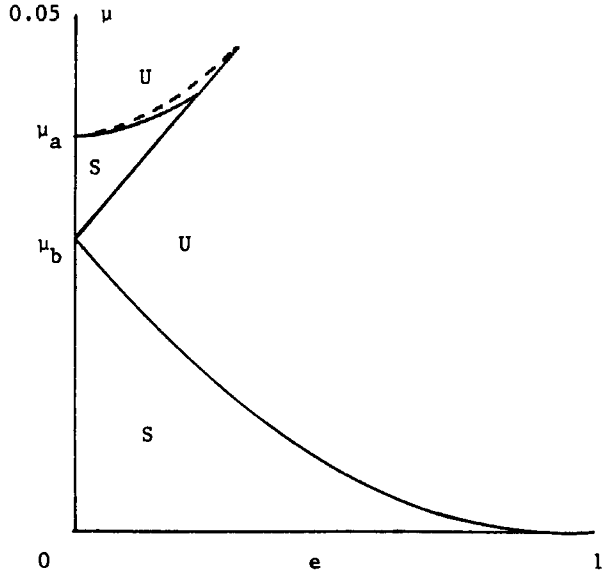

Fig.1. Stability transition curves, S = stable, U = unstable. Danby's [3] numerical results are indistinguishable from eq.(5). However, eq.(4) (solid) differs considerably from Danby's numerical result (dashed).

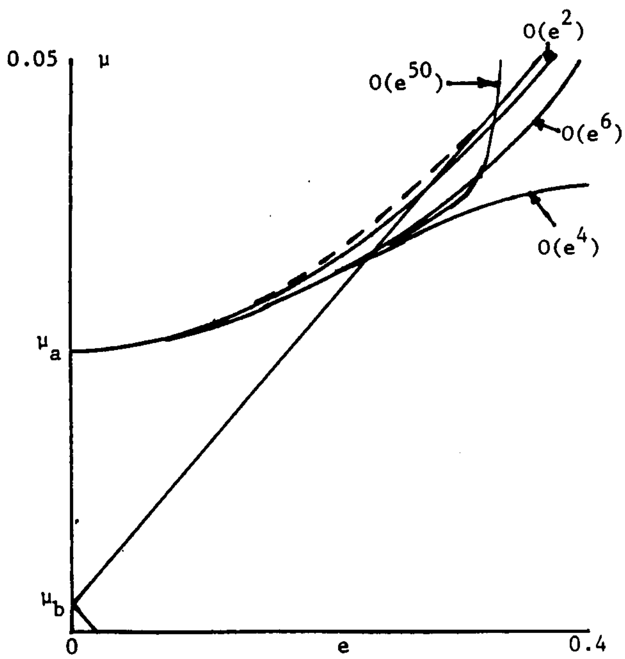

Fig.2. Enlargement of upper left corner of Fig.1 showing partial sums of eq.(4). Dashed line is Danby's numerical result. Note that the $O(e^2)$ result shows better agreement with the numerical result than the higher order approximations. This led Nayfeh and Kamel [5] to write, "Since the second-order expansion reproduces the numerical solution, there is no need to carry out the expansion to higher orders."

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