Reprinted from THE JOURNAL OF THE ACOUSTICAL SOCIETY OF AMERICA, Vol. 44, No. 3, 749–751, September 1968

Torsional Vibrations of Elastic Prolate Spheroids

RICHARD H. RAND

Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14850

Frequency equations and mode shapes are presented in analytic form for the torsional vibrations of solid prolate spheroids and thick prolate spheroidal shells. The solutions are shown to include the previously obtained solid sphere and thin prolate spheroidal shell results as limiting cases. Numerical results are presented for solid prolate spheroids.

INTRODUCTION

THE torsional vibrations of elastic spheres were first studied by Jaerisch[1], and later by Lamb.[2] DiMaggio and Silbiger[3] have studied the torsional vibrations of thin prolate spheriodal shells. Jaerisch[4] presented the governing equations for the torsional vibrations of elastic prolate spheroids, but never solved them.

In this paper, frequency equations and mode shapes are presented in analytic form for the torsional vibrations of solid prolate spheroids and thick prolate spheroidal shells. The solutions are shown to include the previously obtained solid sphere and thin prolate spheroidal shell results as limiting cases. Numerical results are presented for solid prolate spheroids.

I. FORMULATION OF THE PROBLEM

Using Flammer's[5] notation, the prolate spheroidal coordinate system is represented by coordinates η, ξ, ϕ. See Fig. 1.

Let v be the torsional axisymmetric displacement. For free vibrations,

$$v(\eta,\xi,t)=V(\eta,\xi)\exp[i\omega t], \tag{1}$$

where V is the torsional displacement mode and ω is the natural frequency.

For this dilatationless displacement field, the displacement equations of motion of the linear theory of elasticity become[6]

$$(1-\eta^2)^{\frac{1}{2}}\frac{\partial^2}{\partial\eta^2}[(1-\eta^2)^{\frac{1}{2}}V]+(\xi^2-1)^{\frac{1}{2}}\frac{\partial^2}{\partial\xi^2}[(\xi^2-1)^{\frac{1}{2}}V] +c^2(\xi^2-\eta^2)V=0, \tag{2}$$

where

$$c^2=\rho\omega^2d^2/4\mu,$$

ρ is the mass density, μ is the modulus of rigidity, and d is the interfocal distance.

If a prolate spheroidal boundary surface of the form $\xi=a=\text{const.}$, is to be traction free, the stress components $\tau_{\xi\xi}$, $\tau_{\xi\eta}$, $\tau_{\xi\phi}$ must vanish there.

Hooke's law for an elastic solid with a dilatationless displacement field yields

$$\tau_{\xi\xi}=2\mu e_{\xi\xi},$$
$$\tau_{\xi\eta}=2\mu e_{\xi\eta}, \tag{3}$$

and

$$\tau_{\xi\phi}=2\mu e_{\xi\phi},$$

where $e_{\xi\xi}$, $e_{\xi\eta}$, and $e_{\xi\phi}$ are strain components.

The strain-displacement relations (Ref. 6, p. 54) reveal that for the assumed displacement field $e_{\xi\xi}$ and $e_{\xi\eta}$ vanish identically, and that in order for $e_{\xi\phi}$ to vanish on $\xi=a$,

$$\partial V/d\xi-[\xi/(\xi^2-1)]V=0 \quad \text{on} \quad \xi=a. \tag{4}$$

The problem is, then, to find bounded solutions to the partial differential equation, Eq. 2, which satisfy the condition Eq. 4 on the boundaries.

[1] P. Jaerisch, J. F. Math. (Crelle) 88, 131–145 (1880).
[2] H. Lamb, London Math. Soc. Proc. 13, 189–212 (1882).
[3] F. L. DiMaggio and A. Silbiger, J. Acoust. Soc. Am. 33, 56–58 (1961).
[4] P. Jaerisch, J. F. Math (Crelle) 104, 177–210 (1889).
[5] C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, Calif., 1957).
[6] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover Publications, Inc., New York, 1944).

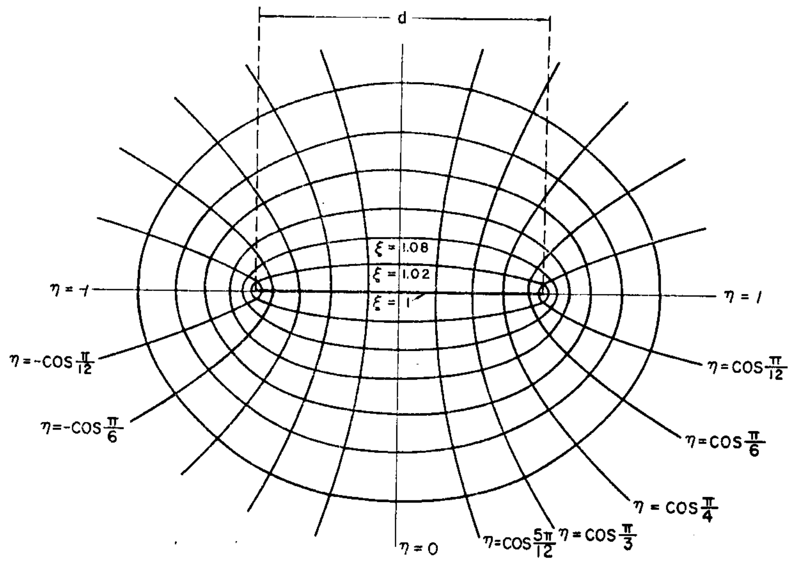

FIG. 1. The prolate spheroidal coordinate system.

II. SOLID PROLATE SPHEROIDS

By separation of the variables, bounded solutions to Eq. ~~1~~ 2 are obtained as

$$V(\eta,\xi)=S_{1n}(c,\eta)R_{1n}^{(1)}(c,\xi), \tag{5}$$

where S_{1n} and $R_{1n}^{(1)}$ are, respectively, the prolate spheroidal angle and radial functions of the first kind, order 1, and degree n.

Substitution of Eq. 5 into Condition 4 yields the frequency equation

$$\frac{dR_{1n}^{(1)}}{d\xi}(c,a)-\frac{a}{a^2-1}R_{1n}^{(1)}(c,a)=0. \tag{6}$$

III. THICK PROLATE SPHEROIDAL SHELLS

Consider a prolate spheroidal shell bounded by confocal spheroids, and thus having a variable thickness. Let $\xi=a_1$ represent the outer boundary and let $\xi=a_2$ represent the inner boundary.

By separation of the variables, bounded solutions to Eq. ~~1~~ 2 are obtained as

$$V(\eta,\xi)=S_{1n}(c,\eta)[R_{1n}^{(1)}(c,\xi)+KR_{1n}^{(2)}(c,\xi)], \tag{7}$$

where $R_{1n}^{(2)}$ is the prolate spheroidal radial function of the second kind, order 1, and degree n, and K is an arbitrary constant.

Define

$$f_i(a_j)=(dR_{1n}^{(i)}/d\xi)(c,a_j)-[a_j/(a_j^2-1)]R_{1n}^{(i)}(c,a_j). \tag{8}$$

Then substitution of Eq. 7 into Condition 4 at $\xi=a_1$ and at $\xi=a_2$, yields, respectively,

$$f_1(a_1)+Kf_2(a_1)=0 \tag{9}$$

and

$$f_1(a_2)+Kf_2(a_2)=0. \tag{10}$$

Solving Eqs. 9 and 10 for K,

$$K=-f_2(a_1)/f_1(a_1)=-f_2(a_2)/f_1(a_2), \tag{11}$$

whence the frequency equation becomes

$$f_1(a_1)f_2(a_2)-f_1(a_2)f_2(a_1)=0. \tag{12}$$

IV. LIMITING CASE OF A SOLID SPHERE

The solution for a solid sphere of radius R may be obtained from Eqs. 5 and 6 by the following limiting process.

Let

$$\xi\rightarrow\infty,\quad d\rightarrow 0 \tag{13}$$

such that

$$\xi d\rightarrow 2r,\quad ad\rightarrow 2R; \tag{14}$$

and let

$$\eta=\cos\theta, \tag{15}$$

where r is the radial spherical coordinate and θ is the polar angle.

Then

$$c^2=\rho\omega^2d^2/4\mu\rightarrow 0, \tag{16}$$

$$\xi^2c^2\rightarrow\rho\omega^2r^2/\mu=z^2, \tag{17}$$

$$a^2c^2\rightarrow\rho\omega^2R^2/\mu=Z^2, \tag{18}$$

and (see Ref. 5)

$$S_{1n}(c,\eta)\rightarrow P_n^1(\eta), \tag{19}$$

$$R_{1n}^{(1)}(c,\xi)\rightarrow j_n(z), \tag{20}$$

where P_n^1 is the associated Legendre function of the first kind, order 1, and degree n, and j_n is the spherical Bessel function of the first kind, of order n.

Equations 5 and 6 become, respectively,

$$V(\eta,r)=P_n^1(\eta)j_n(z) \tag{21}$$

and

$$Z(dj_n/dz)(Z)-j_n(Z)=0. \quad (22)$$

Equations 21 and 22 agree with the results of Jaerisch[1] and Lamb.[2]

V. LIMITING CASE OF A THIN PROLATE SPHEROIDAL SHELL

Let the minimum thickness of the shell be $h \ll d$, and let the middle surface be $\xi=\alpha$. Then the outer boundary becomes (see Ref. 3) $\xi=\alpha+(h/d)$ and the inner boundary $\xi=\alpha-(h/d)$. The frequency equation, Eq. 12, becomes

$$f_1\left(\alpha+\left[\frac{h}{d}\right]\right)f_2\left(\alpha-\left[\frac{h}{d}\right]\right)-f_1\left(\alpha-\left[\frac{h}{d}\right]\right)f_2\left(\alpha+\left[\frac{h}{d}\right]\right)=0. \quad (23)$$

Expanding f_i in a Taylor's series about α, obtain

$$f_i\left(\alpha\pm\left[\frac{h}{d}\right]\right)=f_i(\alpha)\pm\left[\frac{h}{d}\right]\frac{df_i}{d\xi}(\alpha)+\cdots. \quad (24)$$

Substituting Eq. 24 into Eq. 23, and neglecting terms of order $[h/d]^2$ and higher, yields

$$f_1(\alpha)(df_2/d\xi)(\alpha)-f_2(\alpha)(df_1/d\xi)(\alpha)=0. \quad (25)$$

From Eq. 8,

$$\frac{df_i}{d\xi}(\alpha)=\frac{d^2R_{1n}^{(i)}}{d\xi^2}(c,\alpha)-\frac{\alpha}{\alpha^2-1}\frac{dR_{1n}^{(i)}}{d\xi}(c,\alpha)+\frac{\alpha^2+1}{(\alpha^2-1)^2}R_{1n}^{(i)}(c,\alpha). \quad (26)$$

The functions $R_{1n}^{(i)}$ satisfy the differential equation (see Ref. 5)

$$(\xi^2-1)\frac{d^2R_{1n}^{(i)}}{d\xi^2}(c,\xi)+2\xi\frac{dR_{1n}^{(i)}}{d\xi}(c,\xi)-[\lambda_{1n}-c^2\xi^2+(\xi^2-1)^{-1}]R_{1n}^{(i)}(c,\xi)=0, \quad (27)$$

where λ_{1n} is the prolate spheroidal eigenvalue of order 1 and degree n.

Substituting Eqs. 26 and 27 into Eq. 25 yields the frequency equation

$$2-\lambda_{1n}+c^2\alpha^2=0, \quad (28)$$

which agrees with the solution presented by DiMaggio and Silbiger.[3]

TABLE I. Values of c for solid prolate spheroids.

a	β	$n=2$	$n=3$	$n=4$	$n=5$
1.01	7.12	2.56	4.27	5.89	7.49
1.02	5.07	2.53	4.22	5.82	7.39
1.03	4.17	2.51	4.17	5.75	7.29
1.04	3.64	2.48	4.12	5.68	7.20
1.05	3.28	2.46	4.07	5.61	7.11
1.06	3.01	2.43	4.03	5.54	7.02
1.07	2.81	2.41	3.98	5.48	6.93
1.08	2.65	2.38	3.94	5.41	6.85
1.09	2.51	2.36	3.90	5.35	6.77
1.10	2.40	2.34	3.86	5.29	6.69

VI. NUMERICAL RESULTS FOR SOLID PROLATE SPHEROIDS

The frequency equation (Eq. 6) may be solved approximately with the aid of the tabulation of $R_{1n}^{(1)}$ and $dR_{1n}^{(1)}/d\xi$, as presented in Ref. 7.

The results of such computations are shown in Table I, where the lowest values of c satisfying Eq. 6 are tabulated. The parameter β is the ratio of the major to the minor axis of the prolate spheroid,

$$\beta=a/(a^2-1)^{\frac{1}{2}}. \quad (29)$$

ACKNOWLEDGMENTS

The author wishes to thank Professor F. DiMaggio of Columbia University for suggesting this problem. This investigation was partially sponsored by the U. S. Office of Naval Research under contract with Columbia University.

[7] S. Hanish, R. V. Baier, W. H. Buckler, and B. K. Carmichael, "Tables of Spheroidal Wave Functions, Part 2—Prolate Radial Functions of the First and Second Type," Naval Res. Lab. Rept. 6502, Washington D. C. (1967).