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PATTERN FORMATION IN A PARAMETRICALLY-EXCITED P.D.E.

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ABSTRACT

We investigate the dynamics of the parametrically-excited P.D.E.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \varepsilon\beta \frac{\partial u}{\partial t} + (\delta + \varepsilon\gamma \cos t)\, u = \varepsilon\alpha u^3 \quad (1)$$

with Neumann boundary conditions on a rectangular region:

$$\frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0, \pi \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0 \quad \text{for } y = 0, \ \frac{\pi}{\mu}$$

where $0 < \mu \leq 1$. Our approach involves expanding $u(x, y, t)$ in a 3-term Fourier series truncation:

$$u = f_0(t) + f_1(t)\cos x + f_2(t)\cos \mu y \quad (2)$$

By substituting (2) into (1) we obtain a system of 3 coupled nonlinear Mathieu equations which we analyze using averaging in the neighborhood of 2 : 1 resonance.

By varying the parameters c and δ we obtain bifurcation curves which divide the $c\delta$-plane into more than forty regions, each containing a distinct slow flow. Individual regions are found to differ from one another with respect to such features as the number and character of slow flow equilibria, and the presence or absence of a limit cycle. When interpreted in the original variable u, these regions account for a variety of patterns which may be classified as stationary, traveling or rotating.

This type of behavior is comparable to various experimental observations made by other investigators on vertically driven fluids or sand.

1 Introduction

Parametrically-excited O.D.E.'s and P.D.E.'s have found numerous applications in mechanics, *e.g.* dynamic buckling, stability of motion, and water waves in a vertically forced tank (Miles and Henderson, 1990). More recent applications include pattern formation in vertically forced granular layers (Melo *et al.*, 1995), (Rothman, 1998).

Three recent works have treated the dynamics of equation (1) with only one spatial dimension x (Rand, 1996), (Newman *et al.*, 1999), (Armbruster *et al.*, 2001). It was shown that the steady state behavior consists either of (i) the trivial solution $u = 0$; (ii) a single spatial mode of the form $\cos nx$, varying periodically in time; (iii) a more complicated motion consisting of more than one mode, varying periodically or quasiperiodically in time; or (iv) unbounded growth. The question of which of these occurs was shown to depend upon both the parameters as well as the initial conditions.

In this paper we extend these previous works to include two spatial dimensions. The resulting steady states will be shown to consist of patterns. Our treatment closely follows that of (Rand, 1996).

2 Averaging

After substituting (2) into (1), simplifying the expression, and minimizing the error of this projection by making the residual orthogonal to (2) (as is usual in Galerkin projection), we arrive at the following three O.D.E.'s

$$\frac{d^2 f_0}{dt^2} + \varepsilon\beta \frac{d f_0}{dt} + (\delta + \varepsilon\gamma \cos t)\, f_0$$

$$
\begin{aligned}
& - \varepsilon\alpha\left(f_0^3+\frac{3}{2}f_0f_1^2+\frac{3}{2}f_0f_2^2\right)=0\\
\frac{d^2f_1}{dt^2} & + \varepsilon\beta\frac{df_1}{dt}+(c^2+\delta+\varepsilon\gamma\cos t)f_1\\
& - \varepsilon\alpha\left(\frac{3}{4}f_1^3+3f_0^2f_1+\frac{3}{2}f_1f_2^2\right)=0 \qquad (3)\\
\frac{d^2f_2}{dt^2} & + \varepsilon\beta\frac{df_2}{dt}+(\mu^2c^2+\delta+\varepsilon\gamma\cos t)f_2\\
& - \varepsilon\alpha\left(\frac{3}{4}f_2^3+3f_0^2f_2+\frac{3}{2}f_1f_2^2\right)=0
\end{aligned}
$$

These equations are a system of three coupled damped nonlinear Mathieu equations. In the nonlinear, undamped, uncoupled case, the major resonance is the 2:1 resonance (*i.e.*, the forcing frequency is twice the natural frequency) which causes the trivial solution $u=0$ to become unstable. With damping present, this results in attractive periodic motions. For (3), we are interested in exploring the dynamics of the system in the neighborhood of this 2:1 resonance, where we expect to find stable periodic orbits. These periodic orbits will correspond to spatial patterns in (1) which vary periodically in time. In order to perturb off of this resonance, we set

$$\delta=\frac{1}{4}+\delta_1\varepsilon, \quad c=c_1\sqrt{\varepsilon} \qquad (4)$$

so that the natural frequencies of (3) become $\omega_0^2=1/4+\delta_1\varepsilon$, $\omega_1^2=1/4+(c_1^2+\delta_1)\varepsilon$, and $\omega_2^2=1/4+(\mu^2c_1^2+\delta_1)\varepsilon$ respectively. We then use the method of averaging to approximate (3) for small ε. We simplify by neglecting terms of $O(\varepsilon^2)$ after the substitution of (4) into (3), and then set

$$f_i=a_i(t)\cos\frac{t}{2}+b_i(t)\sin\frac{t}{2}, \quad i=0,1,2. \qquad (5)$$

By averaging over one period ($t=\pi$) we then obtain six autonomous first order equations on $a_i(t)$ and $b_i(t)$, namely

$$
\begin{aligned}
4\frac{da_0}{dt} &= -\varepsilon(b_0[4\gamma-8\delta_1]+\alpha[9b_0b_1^2+9b_0b_2^2+6a_0a_1b_1\\
&\quad + 6a_0a_2b_2+6b_0^3+3a_1^2b_0+3a_2^2b_0+6a_0^2b_0])\\
8\frac{da_1}{dt} &= -\varepsilon(b_1[8\gamma-16\delta_1-16c_1^2]+\alpha[18b_1b_2^2+12a_1a_2b_2\\
&\quad + 9b_1^3+36b_0^2b_1+6a_2^2b_1+9a_1^2b_1+12a_0^2b_1+24a_0a_1b_0])\\
8\frac{da_2}{dt} &= -\varepsilon(b_2[8\gamma-16\delta_1-16\mu^2c_1^2]+\alpha[18b_2b_1^2+12a_2a_1b_1\\
&\quad + 9b_2^3+36b_0^2b_2+6a_1^2b_2+9a_2^2b_2+12a_0^2b_2+24a_0a_2b_0])\\
4\frac{db_0}{dt} &= -\varepsilon(a_0[4\gamma+8\delta-\alpha[3a_0b_2^2+6a_2b_0b_2+3a_0b_1^2 \qquad (6)\\
&\quad + 6a_1b_0b_1+6a_0b_0^2+9a_0a_2^2+9a_0a_1^2+6a_0^3])\\
8\frac{db_1}{dt} &= -\varepsilon(a_1[8\gamma+16\delta+16c_1^2]-\alpha[6a_1b_2^2+12a_2b_1b_2\\
&\quad + 9a_1b_1^2+24a_0b_0b_1+12a_1b_0^2+18a_1a_2^2+9a_1^3+36a_0^2a_1])\\
8\frac{db_2}{dt} &= -\varepsilon(a_2[8\gamma+16\delta+16\mu^2c_1^2]-\alpha[6a_2b_1^2+12a_1b_1b_2\\
&\quad + 9a_2b_2^2+24a_0b_0b_2+12a_2b_0^2+18a_2a_1^2+9a_2^3+36a_0^2a_2])
\end{aligned}
$$

where we have set $\beta=0$ in order to simplify the following analysis.

3 Slow Flow Equilibria

Since u in equation (1) may be scaled to absorb α, only the sign of α is significant. We assume $\alpha>0$ and set $\alpha=1$ in what follows for brevity. We also note that since ε in eq.(1) may be scaled to absorb γ, we take $\gamma=1$ in the following.

Equilibria of (6) represent periodic motions of (3) and are termed “modes” (Rand, 1996). These will be denoted by their non-zero components. In order to obtain these equilibria it is necessary to solve 6 simultaneous nonlinear algebraic equations for the unknowns $a_0,\ a_1,\ a_2,\ b_0,\ b_1,\ b_2$. We present an extensive list of such equilibria, which, however, may not be complete. Our purpose is to illustrate that the three term truncation (2) of the P.D.E. has a very complicated bifurcation structure, and the addition of more equilibria not on this list would only increase the complexity of the results. The modes we found in closed-form are

$$
\begin{aligned}
u=0: &\ \{a_{0,1,2}=b_{0,1,2}=0\} && (7)\\
a_0: &\ \left\{a_0^2=\frac{2(1+2\delta_1)}{3},\ a_{1,2}=b_{0,1,2}=0\right\} && (8)\\
b_0: &\ \left\{b_0^2=\frac{-2(1-2\delta_1)}{3},\ a_{0,1,2}=b_{1,2}=0\right\} && (9)\\
a_1: &\ \left\{a_1^2=\frac{8(1+2\delta_1+2c_1^2)}{9},\ a_{0,2}=b_{0,1,2}=0\right\} && (10)\\
a_2: &\ \left\{a_2^2=\frac{8(1+2\delta_1+2c_1^2\mu^2)}{9},\ a_{0,1}=b_{0,1,2}=0\right\} && (11)\\
b_1: &\ \left\{b_1^2=\frac{-8(1-2\delta_1-2c_1^2)}{9},\ a_{0,1,2}=b_{0,2}=0\right\} && (12)\\
b_2: &\ \left\{b_2^2=\frac{-8(1-2\delta_1-2c_1^2\mu^2)}{9},\ a_{0,1,2}=b_{0,2}=0\right\} && (13)\\
a_0a_1: &\ \left\{a_0^2=\frac{2(1+2\delta_1+4c_1^2)}{15},\ a_1^2=\frac{16(1+2\delta_1-c_1^2)}{45},\right.\\
&\ \left. a_2=b_{0,1,2}=0\right\} && (14)\\
a_0a_2: &\ \left\{a_0^2=\frac{2(1+2\delta_1+4c_1^2\mu^2)}{15},\ a_2^2=\frac{16(1+2\delta_1-c_1^2\mu^2)}{45},\right.
\end{aligned}
$$

$$a_1 = b_{0,1,2} = 0\} \tag{15}$$

$$a_0 b_1: \left\{ a_0^2 = \frac{2(5+2\delta_1-4c_1^2)}{3},\; b_1^2 = \frac{-16(1-c_1^2)}{3}, \right.$$
$$a_{1,2} = b_{0,2} = 0\} \tag{16}$$

$$a_0 b_2: \left\{ a_0^2 = \frac{-2(-5-2\delta_1+4c_1^2)}{3},\; b_2^2 = \frac{16(c_1^2\mu^2-1)}{3}, \right.$$
$$a_{0,1,2} = b_{0,1} = 0\} \tag{17}$$

$$b_0 a_1: \left\{ b_0^2 = \frac{-2(5-2\delta_1+4c_1^2)}{3},\; a_1^2 = \frac{16(1+c_1^2)}{3}, \right.$$
$$a_{0,2} = b_{1,2} = 0\} \tag{18}$$

$$b_0 a_2: \left\{ b_0^2 = \frac{-2(5-2\delta_1+4c_1^2)}{3},\; a_2^2 = \frac{16(1+c_1^2\mu^2)}{3}, \right.$$
$$a_{0,1} = b_{1,2} = 0\} \tag{19}$$

$$b_0 b_1: \left\{ b_0^2 = \frac{-2(1-2\delta_1-4c_1^2)}{15},\; b_1^2 = \frac{-16(1-2\delta_1+c_1^1)}{45}, \right.$$
$$a_{0,1,2} = b_2 = 0\} \tag{20}$$

$$b_0 b_2: \left\{ b_0^2 = \frac{-2(1-2\delta_1-4c_1^2)}{15},\; b_2^2 = \frac{-16(1-2\delta_1+c_1^2\mu^2)}{45}, \right.$$
$$a_{0,1,2} = b_1 = 0\} \tag{21}$$

$$a_1 a_2: \left\{ a_1^2 = \frac{8(1+2\delta_1-2c_1^2+4c_1^2\mu^2)}{27}, \right. \tag{22}$$
$$\left. a_2^2 = \frac{8(1+2\delta_1+4c_1^2-2c_1^2\mu^2)}{27},\; a_0 = b_{0,1,2} = 0 \right\}$$

$$a_1 b_2: \left\{ a_1^2 = \frac{4(10+4\delta_1+12c_1^2-8c_1^2\mu^2)}{15}, \right. \tag{23}$$
$$\left. b_2^2 = \frac{-4(10-4\delta_1+8c_1^2-12c_1^2\mu^2)}{15},\; a_{0,2} = b_{0,1} = 0 \right\}$$

$$b_1 a_2: \left\{ b_1^2 = \frac{-4(10-4\delta_1-12c_1^2+8c_1^2\mu^2)}{15}, \right. \tag{24}$$
$$\left. a_2^2 = \frac{4(10+4\delta_1-8c_1^2+12c_1^2\mu^2)}{15},\; a_{0,1} = b_{0,2} = 0 \right\}$$

$$b_1 b_2: \left\{ b_1^2 = \frac{8(-1+2\delta_1-2c_1^2+4c_1^2\mu^2)}{27}, \right. \tag{25}$$
$$\left. b_2^2 = \frac{-8(1-2\delta_1-4c_1^2+c_1^2\mu^2)}{27},\; a_{0,1,2} = b_0 = 0 \right\}$$

$$a_0 a_1 a_2: \left\{ a_0^2 = \frac{2(1+2\delta_1+4c_1^2+4c_1^2\mu^2)}{27}, \right. \tag{26}$$
$$a_1^2 = \frac{8(2+4\delta_1-10c_1^2+8c_1^2\mu^2)}{81},$$
$$\left. a_2^2 = \frac{8(2+4\delta_1+8c_1^2-10c_1^2\mu^2)}{81},\; b_{0,1,2} = 0 \right\}$$

$$a_0 a_1 b_2: \left\{ a_0^2 = \frac{2(15+6\delta_1+28c_1^2-12c_1^2\mu^2)}{93}, \right. \tag{27}$$
$$a_1^2 = \frac{4(20+8\delta_1-4c_1^2-16c_1^2\mu^2)}{93},$$
$$\left. b_2^2 = \frac{16(-11+8\delta_1-4c_1^2+15c_1^2\mu^2)}{93},\; b_{0,1} = a_2 = 0 \right\}$$

$$a_0 b_1 a_2: \left\{ a_0^2 = \frac{2(15+6\delta_1+28c_1^2-12c_1^2\mu^2)}{93}, \right. \tag{28}$$
$$a_2^2 = \frac{4(20+8\delta_1-16c_1^2-4c_1^2\mu^2)}{93},$$
$$\left. b_1^2 = \frac{16(-11+8\delta_1+15c_1^2-4c_1^2\mu^2)}{93},\; b_{0,1} = a_2 = 0 \right\}$$

$$a_0 b_1 b_2: \left\{ a_0^2 = \frac{2(13+10\delta_1-4c_1^2-4c_1^2\mu^2)}{15}, \right. \tag{29}$$
$$b_1^2 = \frac{-16(3+c_1^2-4c_1^2\mu^2)}{45},$$
$$\left. b_2^2 = \frac{-16(3-4c_1^2+c_1^2\mu^2)}{45},\; a_{1,2} = b_0 = 0 \right\}$$

$$b_0 a_1 a_2: \left\{ b_0^2 = \frac{-4(6-5\delta_1+2c_1^2+4c_1^2\mu^2)}{15}, \right. \tag{30}$$
$$a_1^2 = \frac{16(3-c_1^2+4c_1^2\mu^2)}{45},$$
$$\left. a_2^2 = \frac{16(3+4c_1^2-c_1^2\mu^2)}{45},\; a_0 = b_{1,2} = 0 \right\}$$

$$b_0 a_1 b_2: \left\{ b_0^2 = \frac{2(-15+6\delta_1-12c_1^2+28c_1^2\mu^2)}{93}, \right. \tag{31}$$
$$a_1^2 = \frac{-16(-11-8\delta_1-15c_1^2+4c_1^2\mu^2)}{93},$$
$$\left. b_2^2 = \frac{-16(5-2\delta_1+4c_1^2+c_1^2\mu^2)}{93},\; a_{0,2} = b_1 = 0 \right\}$$

$$b_0 b_1 a_2: \left\{ b_0^2 = \frac{-2(15-6\delta_1-28c_1^2+12c_1^2\mu^2)}{93}, \right. \tag{32}$$
$$b_1^2 = \frac{-16(5-2\delta_1+c_1^2+4c_1^2\mu^2)}{93},$$
$$\left. a_2^2 = \frac{16(11+8\delta_1-4c_1^2+15c_1^2\mu^2)}{93},\; a_{0,1} = b_2 = 0 \right\}$$

$$b_0 b_1 b_2: \left\{ b_0^2 = \frac{2(-1+2\delta_1+4c_1^2+4c_1^2\mu^2)}{27}, \right. \tag{33}$$
$$b_1^2 = \frac{16(-1+2\delta_1-5c_1^2+4c_1^2\mu^2)}{81},$$
$$\left. b_2^2 = \frac{-16(1-2\delta_1-4c_1^2+5c_1^2\mu^2)}{81},\; a_{0,1,2} = 0 \right\}$$

4 Conditions for Existence of the Modes

The conditions which determine the existence of the solutions in equations (7)–(33) may be found by requiring that $a_i,\; b_i$

be real. This yields, for each mode, a set of inequalities which when satisfied by the parameters indicate the existence of that mode. These are found to be

$$u=0:\ \text{always exists} \tag{34}$$

$$a_0:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2} \tag{35}$$

$$b_0:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2} \tag{36}$$

$$a_1:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}-c^2 \tag{37}$$

$$b_1:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}-c^2 \tag{38}$$

$$a_2:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}-c^2\mu^2 \tag{39}$$

$$b_2:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}-c^2\mu^2 \tag{40}$$

$$a_0a_1:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2}{2} \tag{41}$$

$$a_0a_2:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2\mu^2}{2} \tag{42}$$

$$a_0b_1:\ \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+2c^2,\ \text{and}\ c^2>\varepsilon \tag{43}$$

$$a_0b_2:\ \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+2c^2\mu^2,\ \text{and}\ c^2\mu^2>\varepsilon \tag{44}$$

$$b_0a_1:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2 \tag{45}$$

$$b_0a_2:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2\mu^2 \tag{46}$$

$$b_0b_1:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}+\frac{c^2}{2} \tag{47}$$

$$b_0b_2:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}+\frac{c^2\mu^2}{2} \tag{48}$$

$$a_1a_2:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}+c^2-2c^2\mu^2 \tag{49}$$

$$a_1b_2:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2-3c^2\mu^2,\ \text{and}\quad \delta > \frac{1}{4}-\frac{5\varepsilon}{2}-3c^2+2c^2\mu^2 \tag{50}$$

$$b_1a_2:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}-3c^2+2c^2\mu^2,\ \text{and}\quad \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+2c^2-3c^2\mu^2 \tag{51}$$

$$b_1b_2:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}+c^2-2c^2\mu^2 \tag{52}$$

$$a_0a_1a_2:\ \delta > \frac{1}{4}-\frac{\varepsilon}{2}+\frac{5c^2}{2}-2c^2\mu^2 \tag{53}$$

$$a_0a_1b_2:\ \delta > \frac{1}{4}+\frac{11\varepsilon}{8}+\frac{c^2}{2}-\frac{15c^2\mu^2}{8},\ \text{and}\quad \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+\frac{c^2}{2}+2c^2\mu^2 \tag{54}$$

$$a_0b_1a_2:\ \delta > \frac{1}{4}+\frac{11\varepsilon}{8}-\frac{15c^2}{8}+\frac{c^2\mu^2}{2},\ \text{and}\quad \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+2c^2+\frac{c^2\mu^2}{2} \tag{55}$$

$$a_0b_1b_2:\ \delta > \frac{1}{4}-\frac{13\varepsilon}{10}+\frac{2c^2}{5}+\frac{2c^2\mu^2}{5},\ \text{and}\quad 4c^2\mu^2-c^2>3\varepsilon \tag{56}$$

$$b_0a_1a_2:\ \delta > \frac{1}{4}+\frac{13\varepsilon}{10}+\frac{2c^2}{5}+\frac{2c^2\mu^2}{5},\ \text{and}\quad c^2-4c^2\mu^2<3\varepsilon \tag{57}$$

$$b_0a_1b_2:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2+\frac{c^2\mu^2}{2} \tag{58}$$

$$b_0b_1a_2:\ \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+\frac{c^2}{2}+2c^2\mu^2 \tag{59}$$

$$b_0b_1b_2:\ \delta > \frac{1}{4}+\frac{\varepsilon}{2}+\frac{5c^2}{2}-2c^2\mu^2 \tag{60}$$

5 Conditions for Stability of the Modes

The conditions which determine the stability of the slow flow equilibria in equations (7)–(33) can be found by linearizing (6) about each equilibrium point and requiring that all eigenvalues of the linear system be pure imaginary. Since there is no asymptotic stability possible, *i.e.*, we have set $\beta=0$, this procedure will determine which equilibrium points in the slow-flow equations are centers, and hence, because of their structural instability, will become stable spirals when the damping is small but non-zero.

After linearization of the slow-flow equations, the characteristic equation whose roots are the eigenvalues of the Jacobian is always of the general form:

$$\lambda^6+A\lambda^4+B\lambda^2+C=0 \tag{61}$$

where A,B,C are polynomial functions of $\delta,\varepsilon,c,$ and μ. The necessary and sufficient conditions for this equation to have six pure imaginary roots are

$$A>0,\ B>0,\ C>0,\ \text{and} \tag{62}$$

$$-27\left(C-\frac{AB}{3}+\frac{2A^3}{27}\right)^2-4\left(B-\frac{A^2}{3}\right)^3>0, \tag{63}$$

which can been derived by considering (61) to be a cubic equation in λ^2 (Birkhoff and MacLane, 1965). These criteria indicate that the respective modes are stable in the following regions of parameter space

$$u=0: \; \delta > \frac{1}{4}+\frac{\varepsilon}{2}; \tag{64}$$

$$\delta < \frac{1}{4}-\frac{\varepsilon}{2}-c^2; \tag{65}$$

$$\delta > \frac{1}{4}+\frac{\varepsilon}{2}-c^2\mu^2, \;\; \delta < \frac{1}{4}-\frac{\varepsilon}{2}; \tag{66}$$

$$\delta > \frac{1}{4}-\frac{\varepsilon}{2}-c^2, \;\; \delta < \frac{1}{4}-\frac{\varepsilon}{2}-c^2\mu^2 \tag{67}$$

$$a_0: \; \delta > \frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2}{2}, \;\; \varepsilon > c^2; \tag{68}$$

$$\frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2}{2} < \delta < \frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2\mu^2}{2}, \tag{69}$$
$$c^2\mu^2 < \varepsilon < c^2;$$

$$\frac{1}{4}-\frac{\varepsilon}{2} < \delta < \frac{1}{4}-\frac{\varepsilon}{2}+\frac{c^2\mu^2}{2}, \;\; \varepsilon < c^2\mu^2; \tag{70}$$

$b_0:$ always unstable (71)

$$a_1: \; \delta > \frac{1}{4}-\frac{\varepsilon}{2}-2c^2+c^2\mu^2, \tag{72}$$
$$\delta < \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2-3c^2\mu^2$$

$b_1:$ always unstable (73)

$$a_2: \; \delta > \frac{1}{4}-\frac{\varepsilon}{2}+c^2-2c^2\mu^2, \tag{74}$$
$$\delta < \frac{1}{4}+\frac{5\varepsilon}{2}-3c^2+2c^2\mu^2;$$

$$\delta > \frac{1}{4}+\frac{5\varepsilon}{2}-3c^2+2c^2\mu^2, \tag{75}$$
$$\delta < \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2\mu^2,$$
$$\delta < \frac{1}{4}-\frac{\varepsilon}{2}+c^2-2c^2\mu^2$$

$b_2:$ always unstable (76)

$$a_0a_1: \; \frac{1}{4}+\frac{4\varepsilon}{7}-\frac{4c^2}{7} < \delta < \frac{1}{4}+\frac{11\varepsilon}{8}+\frac{c^2}{2}-\frac{15c^2\mu^2}{8}, \tag{77}$$
$$\begin{aligned}&33124\varepsilon^4+112112\delta\varepsilon^3+72384c^2\varepsilon^3-28028\varepsilon^3\\&+54096\delta^2\varepsilon^2+61344c^2\delta\varepsilon^2-27048\delta\varepsilon^2\\&-140096c^4\varepsilon^2-15336c^2\varepsilon^2+3381\varepsilon^2\\&-68992\delta^3\varepsilon-148032c^2\delta^2\varepsilon+51744\delta^2\varepsilon\\&+2176c^4\delta\varepsilon+74016c^2\delta\varepsilon-12936\delta\varepsilon\\&+44544c^6\varepsilon-544c^4\varepsilon-9252c^2\varepsilon\\&+1078\varepsilon+12544\delta^4+37632c^2\delta^3-12544\delta^3\\&+3136c^4\delta^2-28224c^2\delta^2+4704\delta^2-37632c^6\delta\\&-1568c^4\delta+7056c^2\delta-784\delta+12544c^8+9408c^6\\&+196c^4-588c^2+49>0\end{aligned}$$

Note, the notation $\delta > (<)x, \;\; \delta < (>)y$ denotes that $\delta > x, \;\; \delta < y$ and $\delta < x, \;\; \delta > y$ are both valid regions of stability. Clearly, given a particular set of parameter values, at most one of these may be satisfied.

$$a_0a_2: \; \delta > (<)\frac{1}{4}-\frac{\varepsilon}{2}+\frac{5c^2}{2}-2c^2\mu^2, \tag{78}$$
$$\delta < (>)\frac{1}{4}+\frac{11\varepsilon}{8}-\frac{15c^2}{8}+\frac{c^2\mu^2}{2},$$
$$\delta > \frac{1}{4}+\frac{4\varepsilon}{7}-\frac{4c^2\mu^2}{7},$$
$$\begin{aligned}&12544c^8\mu^8+44544c^6\varepsilon\mu^6-37632c^6\delta\mu^6\\&+9408c^6\mu^6-140096c^4\varepsilon^2\mu^4+2176c^4\delta\varepsilon\mu^4\\&-544c^4\varepsilon\mu^4+3136c^4\delta^2\mu^4-1568c^4\delta\mu^4\\&+196c^4\mu^4+72384c^2\varepsilon^3\mu^2+61344c^2\delta\varepsilon^2\mu^2\\&-15336c^2\varepsilon^2\mu^2-148032c^2\delta^2\varepsilon\mu^2\\&+74016c^2\delta\varepsilon\mu^2-9252c^2\varepsilon\mu^2+37632c^2\delta^3\mu^2\\&-28224c^2\delta^2\mu^2+7056c^2\delta\mu^2-588c^2\mu^2\\&+33124\varepsilon^4+112112\delta\varepsilon^3-28028\varepsilon^3\\&+54096\delta^2\varepsilon^2-27048\delta\varepsilon^2+3381\varepsilon^2-68992\delta^3\varepsilon\\&+51744\delta^2\varepsilon-12936\delta\varepsilon+1078\varepsilon+12544\delta^4\\&-12544\delta^3+4704\delta^2-784\delta+49>0\end{aligned}$$

$$a_0b_1: \; \delta > \frac{1}{4}-\frac{5\varepsilon}{2}+2c^2+\frac{c^2\mu^2}{2}, \; \delta > \frac{1}{4}-4\varepsilon+4c^2 \tag{79}$$

$$a_0b_2: \; 4c^2\mu^2-c^2 > (<)3\varepsilon, \tag{80}$$
$$\delta > (<)\frac{1}{4}-\frac{5\varepsilon}{2}+\frac{c^2}{2}+2c^2\mu^2,$$
$$\delta > \frac{1}{4}-4\varepsilon+4c^2\mu^2$$

$$b_0a_1: \; \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2+\frac{c^2\mu^2}{2} < \delta < \frac{1}{4}+4\varepsilon+4c^2 \tag{81}$$

$$b_0a_2: \; c^2-4c^2\mu^2 > (<)3\varepsilon, \tag{82}$$
$$\delta < (>)\frac{1}{4}+\frac{5\varepsilon}{2}+\frac{c^2}{2}+2c^2\mu^2,$$
$$\delta < \frac{1}{4}+4\varepsilon+4c^2\mu^2$$

$b_0b_1:$ always unstable (83)

$b_0b_2:$ always unstable (84)

$$a_1a_2: \; \delta > \frac{1}{4}+\frac{5\varepsilon}{8}-\frac{c^2}{2}-\frac{c^2\mu^2}{2}, \tag{85}$$
$$\delta < \frac{1}{4}+\frac{13\varepsilon}{10}+\frac{2c^2}{5}+\frac{2c^2\mu^2}{5}$$

$$a_1b_2: \; \delta > \frac{1}{4}+\frac{5\varepsilon}{2}+2c^2-3c^2\mu^2, \tag{86}$$
$$\delta > \frac{1}{4}-\frac{5\varepsilon}{2}-3c^2+2c^2\mu^2$$

$$b_1a_2: \; 4c^2 - 4c^2\mu^2 < 3\varepsilon \tag{87}$$

$$b_1b_2: \; \text{always unstable} \tag{88}$$

For the triple-modes, the stability conditions turn out to be very large, and hence we only examine the special case $\mu = 1$ in order to simplify the expressions. They are found to be

$$a_0a_1a_2: \; \delta > \frac{1}{4} + \frac{8\varepsilon}{11} - \frac{8c^2}{11}, \tag{89}$$

$$\begin{aligned}
&302500\,\varepsilon^4 + 1113200\,\delta\varepsilon^3 + 1097600\,c^2\,\varepsilon^3 \\
&-278300\,\varepsilon^3 + 830544\,\delta^2\,\varepsilon^2 + 1341984\,c^2\,\delta\varepsilon^2 \\
&-415272\,\delta\varepsilon^2 - 1990272\,c^4\,\varepsilon^2 - 335496\,c^2\,\varepsilon^2 \\
&+51909\,\varepsilon^2 - 356224\,\delta^3\,\varepsilon - 1598016\,c^2\,\delta^2\,\varepsilon \\
&+267168\,\delta^2\,\varepsilon - 516864\,c^4\,\delta\varepsilon + 799008\,c^2\,\delta\varepsilon \\
&-66792\,\delta\varepsilon + 702464\,c^6\,\varepsilon + 129216\,c^4\,\varepsilon \\
&-99876\,c^2\,\varepsilon + 5566\,\varepsilon + 30976\,\delta^4 + 216832\,c^2\,\delta^3 \\
&-30976\,\delta^3 + 255552\,c^4\,\delta^2 - 162624\,c^2\,\delta^2 + 11616\,\delta^2 \\
&-433664\,c^6\,\delta - 127776\,c^4\,\delta + 40656\,c^2\,\delta \\
&-1936\,\delta + 123904\,c^8 + 108416\,c^6 + 15972\,c^4 \\
&-3388\,c^2 + 121 > 0
\end{aligned}$$

$$a_0a_1b_2: \; \delta < \frac{1}{4} + \frac{8\varepsilon}{3} - \frac{8c^2}{3} \tag{90}$$

$$a_0b_1a_2: \; \delta < \frac{1}{4} + \frac{8\varepsilon}{3} - \frac{8c^2}{3} \tag{91}$$

$$a_0b_1b_2: \; \text{always unstable} \tag{92}$$

$$b_0a_1a_2: \; \delta < \frac{1}{4} + \frac{8\varepsilon}{5} + \frac{8c^2}{5} \tag{93}$$

$$b_0a_1b_2: \; \text{always unstable} \tag{94}$$

$$b_0b_1a_2: \; \text{always unstable} \tag{95}$$

$$b_0b_1b_2: \; \text{always unstable} \tag{96}$$

The regions of stability in (64)–(96) are found to divide the $c\delta$-plane into more than forty regions. These are shown in Fig.1, and are tabulated in Table 1 for the specific case of a square domain (*i.e.*, $\mu = 1$), and the parameter values $\alpha = 1,\ \beta = 0,\ \gamma = 1$, and $\varepsilon = 0.1$. All changes in stability correspond either to a pair of zero eigenvalues, in which case the associated bifurcation is a saddle-center, or to a pair of repeated pure imaginary eigenvalues, in which case the bifurcation is a Hamiltonian-Hopf (van der Meer, 1985).

6 Conclusions

Inspection of Fig.1 shows that the dynamics of the averaged three-term-truncation in equations (3) is very complicated. In most regions shown there is more than one stable steady state, implying that the long-time behavior is dependent on initial conditions. As the number of terms in the truncation (2) is increased, the complexity of the corresponding Fig.1 will increase. This is apparent from a comparison of the results obtained in this paper with those obtained in (Rand, 1996), where only the first two terms of equation (2) were taken, resulting in 18 separate regions of Fig.1, instead of the 49 regions obtained here by including one additional term. Of course the P.D.E. (1) is expected to possess a diagram *infinitely* more complicated than Fig.1. It is likely that there will be infinitely many comparable regions, each with many, perhaps infinitely many, steady states.

Numerical integration of the slow flow eqs.(6) has confirmed the stable steady states of Fig.1, as listed in Table 1.

The various steady state modes exhibited by the averaged equations (6) may be viewed as a surface $u = u(x,y,t)$ moving in time via the ansatz (2). By coloring the values of u differently, these may be viewed as patterns, as shown in Fig.2. The perturbation assumptions made in our treatment restrict the steady state behavior of the coefficients $f_i(t)$ in equation (2) to be periodic with twice the period of the forcing function, *cf.* equation (5). This means that our approximation yields steady state patterns $u(x,y,t)$ which vary periodically in time. Examination of the various resulting steady state patterns allows us to classify them as either stationary, traveling, or rotating, see Fig.2.

Recall that we set the damping constant $\beta = 0$ in order to simplify the analysis. In the presence of damping, $\beta > 0$, equations (6) may also exhibit stable limit cycle solutions. These will correspond to quasiperiodic motions in equations (3), and to quasiperiodic patterns. Although the present paper has not investigated such motions, numerical integration of equations (6) has shown them to exist, as expected from structural stability and generic bifurcation considerations.

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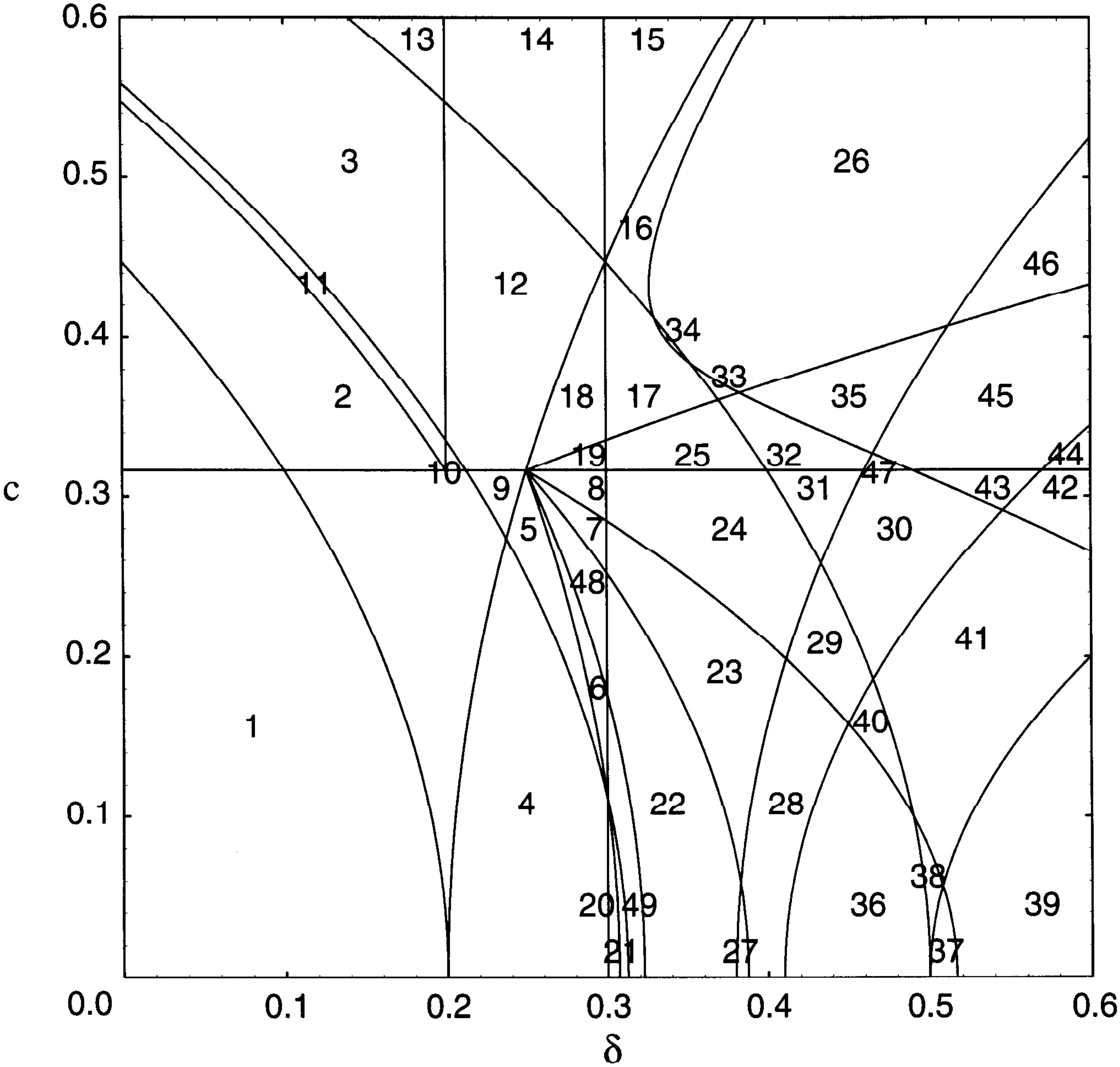

Figure 1. Regions of stable steady state motions of the averaged equations (6). The parameters used are: $\alpha=1$, $\beta=0$, $\gamma=1$, $\varepsilon=0.1$, and $\mu=1$.

region	$u=0$	a_0	a_1 a_2	a_0a_1 a_0a_2	a_0b_1 a_0b_2	b_0a_1 b_0a_2	a_1a_2	a_1b_2 b_1a_2	$a_0a_1a_2$	$a_0a_1b_2$ $a_0b_1a_2$	$b_0a_1a_2$
1	x										
2			x								
3	x		x				x				
4		x	x								
5		x	x				x				
6		x	x	x			x				
7		x	x				x		x	x	
8		x	x				x		x		
9			x				x				
10		x	x								
11	x		x								
12		x	x				x				
13	x						x	x			
14		x					x	x			
15	x	x					x	x			
16	x						x	x	x		
17	x		x				x		x		
18			x				x		x		
19			x		x		x		x		
20			x								
21	x	x	x	x							
22	x	x	x	x			x		x		
23	x	x	x				x		x	x	
24	x	x	x				x		x		
25	x		x		x		x		x		
26	x						x	x			
27	x	x	x	x					x		x
28	x	x	x						x	x	x
29	x	x	x						x		x
30	x	x						x	x		x
31	x	x					x	x	x		
32	x				x		x	x	x		
33	x						x	x	x		
34	x		x				x	x	x		
35	x				x		x	x			
36	x	x	x						x	x	
37	x	x	x			x		x	x	x	
38	x	x	x					x	x	x	
39	x	x				x		x	x		
40	x	x	x						x		
41	x	x						x	x		
42	x	x						x			
43	x	x						x			x
44	x				x			x			
45	x				x			x			x
46	x							x			x
47	x				x			x	x		x
48		x	x	x			x		x		
49	x	x	x	x			x				

Table 1. Stable steady states corresponding to Figure (1).

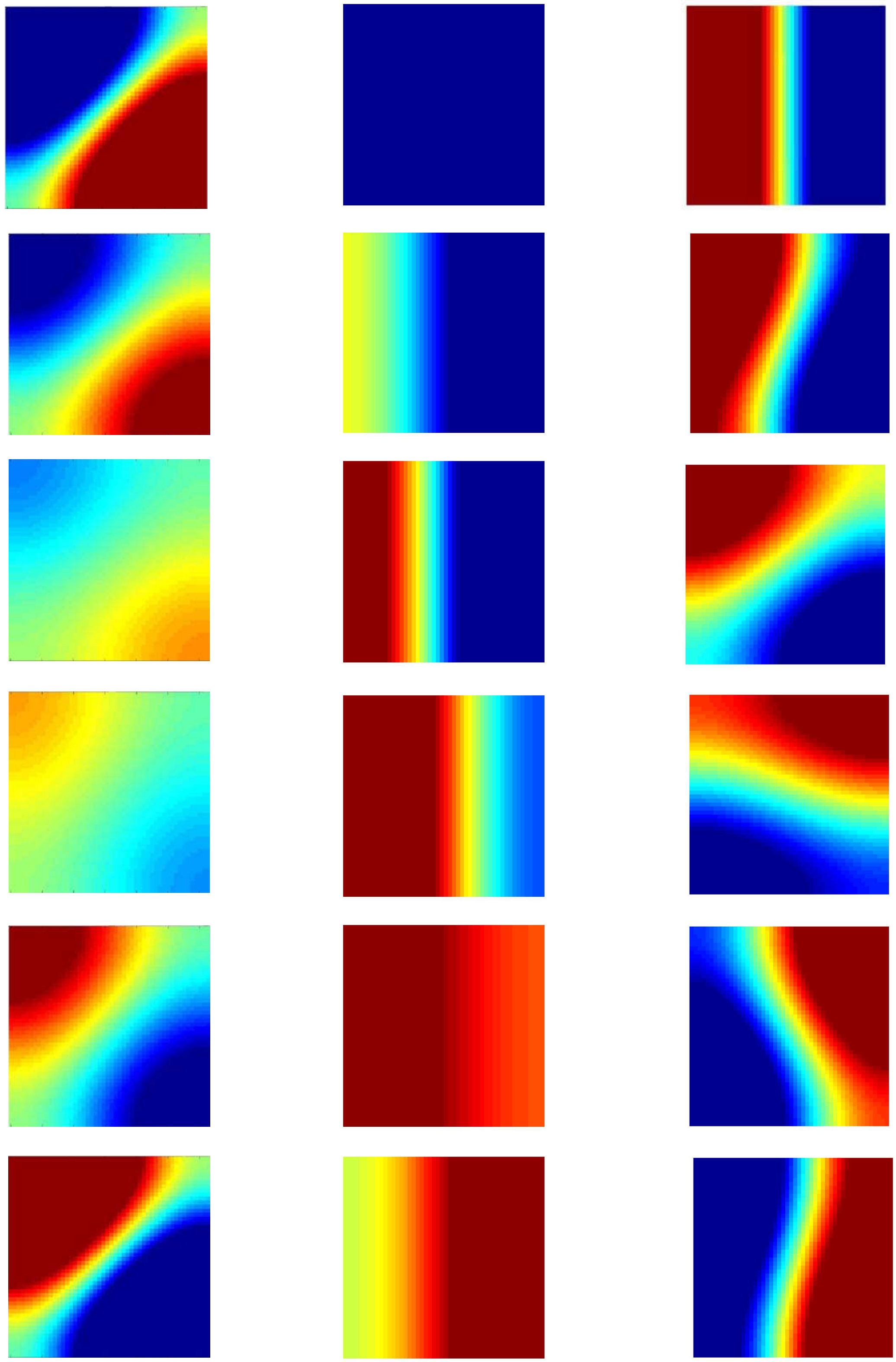

Figure 2. Representative patterns obtained by numerically integrating the slow flow equations (6).

Left: A "stationary" pattern, the $a_1 a_2$ mode.

Center: A "traveling" pattern, the $a_0 b_1$ mode.

Right: A "rotating" pattern, the $a_1 b_2$ mode.