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DYNAMICS OF TWO STRONGLY COUPLED RELAXATION OSCILLATORS*

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Abstract. This paper concerns the dynamics of a pair of identical, linearly coupled van der Pol relaxation oscillators. We study the stability of the in-phase and out-of-phase modes of vibration. The stability of both modes is shown to be governed by the behavior of a linear variational equation with periodic coefficients. Approximate analytical solutions are obtained by the method of matched asymptotic expansions. These analytical results are supplemented by numerical integrations based on Floquet theory.

It is shown that previous work based on the sinusoidal (nonrelaxation) limit fails to predict a significant region of instability for both modes.

Key words. relaxation oscillations, singular perturbations, stability of motion

1. Introduction. In this paper we study the stability of the in-phase and out-of-phase modes of a pair of identical van der Pol relaxation oscillators with strong linear "diffusive" coupling. The van der Pol oscillator is a nonlinear, nonconservative limit cycle oscillator which is described by the equation

$$x''-\nu(1-x^2)x'+x=0 \tag{1}$$

where primes represent differentiation with respect to the time variable $\underline{t}$ and ν is a parameter which measures the strength of the nonlinear damping term. LaSalle [10] and Stoker [16] have shown that this equation possesses a unique stable limit cycle in its phase plane for all positive values of ν and thus exhibits self-sustained periodic oscillations. However, the shape of the limit cycle and the nature of the oscillations depend heavily on ν.

For $\nu \ll 1$, Rayleigh [15] used harmonic balance to show that the limit cycle oscillations are nearly sinusoidal with amplitude 2, i.e.

$$x=2\cos \underline{t}+O(\nu). \tag{2}$$

This will be referred to as the sinusoidal limit of the van der Pol oscillator.

For $\nu \gg 1$, perturbation methods can again be used to analyze the equation. Following Cole [8], we first perform the transformation from $\underline{t}$, ν to t, ε according to

$$t=\nu\underline{t}, \qquad \varepsilon=1/\nu^2 \tag{3}$$

so that (1) becomes

$$\varepsilon\ddot{x}-(1-x^2)\dot{x}+x=0. \tag{4}$$

Dots represent differentiation with respect to the rescaled time variable t and $\nu \gg 1$ corresponds to $\varepsilon \ll 1$ which now serves as the small parameter in the perturbation analysis. This is the van der Pol relaxation oscillator. Andronov and Chaikin [2] used phase plane techniques to find a first approximation to the resulting limit cycle which consists of periods of slow decay followed by rapid jumps. Higher order results were developed later by Dorodnitsyn [9], Carrier [5], and Cole [8] using boundary layer theory and the method of matched asymptotic expansions. Our analysis will be closely related to that of Carrier and Cole.

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In many biological applications, the system of interest consists not of one, but of several relaxation oscillators which are coupled together. For example, van der Pol and van der Mark [20] modeled the heart as two relaxation oscillators interacting through neural and muscular connections. Rand et al. [13], [14] described stomates in green plants as relaxation oscillators coupled through vascular water flow. (See Pavlidis [12] for more biological examples.) In such systems, a most important question concerning the dynamics of the system is that of phase-locking, i.e. under what conditions will the coupling cause the oscillators to lock together and maintain a constant phase difference?

This is the question which we will address while studying a system of two van der Pol relaxation oscillators with linear diffusive coupling given by

$$\begin{aligned} \varepsilon\ddot{x}-(1-x^2)\dot{x}+x&=\alpha(y-x)+\beta(\dot{y}-\dot{x}),\\ \varepsilon\ddot{y}-(1-y^2)\dot{y}+(1+\Delta)y&=\alpha(x-y)+\beta(\dot{x}-\dot{y}). \end{aligned} \tag{5}$$

We restrict our attention to the case of identical oscillators ($\Delta=0$) and concentrate on the case of displacement coupling ($\beta=0$) to complement the work of Belair and Holmes [4] who treated the case of velocity coupling ($\alpha=0$). Our results will also complement those of our previous paper [18] where we examined two van der Pol oscillators with the same coupling but in the sinusoidal limit ($\varepsilon \gg 1$, $\nu \ll 1$). We will see that our present results can be consistently tied together with those of the two earlier papers via numerical investigations.

2. In-phase mode: Variational equations and Floquet theory. To begin our study, we wish to make use of our knowledge of the behavior of a single van der Pol oscillator. We know that (4) exhibits a stable limit cycle solution, which we will call $u(t)$, for all positive values of the parameter ε. Thus when $\Delta=0$, (5) has the exact solution $x(t)\equiv y(t)\equiv u(t)$ which causes the right-hand side to vanish. This solution is the in-phase mode which exists for all values of the coupling parameter α. To determine its stability, we examine the stability of the origin in the phase space of the associated linear variational equations. Hence, let $x=u+\xi$, $y=u+\eta$, and substitute into (5) with $\beta=\Delta=0$. Retaining linear terms in the variations ξ, η and their derivatives, we obtain the linear variational equations

$$\varepsilon\ddot{\xi}-(1-u^2)\dot{\xi}+(1+2u\dot{u})\xi=\alpha(\eta-\xi), \tag{6}$$

$$\varepsilon\ddot{\eta}-(1-u^2)\dot{\eta}+(1+2u\dot{u})\eta=\alpha(\xi-\eta). \tag{7}$$

To uncouple these equations, we add (7) to (6), subtract (7) from (6) and let $\dot{h}=\xi+\eta$, $\dot{v}=\xi-\eta$. Integrating once then yields

$$\varepsilon\ddot{h}-(1-u^2)\dot{h}+h=k_1, \tag{8}$$

$$\varepsilon\ddot{v}-(1-u^2)\dot{v}+pv=k_2, \qquad p=1+2\alpha, \tag{9}$$

where k_1 and k_2 are constants of integration. The particular solutions $h_p=k_1$, $v_p=k_2/p$ are constants and do not contribute to the variations given by

$$\xi=(\dot{h}+\dot{v})/2, \qquad \eta=(\dot{h}-\dot{v})/2. \tag{10}$$

Hence we study the homogeneous equations

$$\varepsilon\ddot{h}-(1-u^2)\dot{h}+h=0, \tag{11}$$

$$\varepsilon\ddot{v}-(1-u^2)\dot{v}+pv=0. \tag{12}$$

Note that Floquet theory (Coddington and Levinson [7]) can be applied to this system of second-order linear ordinary differential equations with periodic coefficients, the periodicity arising from the term $1-u^2$ which has period T (equal to half the period of $u(t)$ due to symmetry). Consider a fundamental solution matrix of (11) or (12):

$$\mathbf{W}(t)=\begin{bmatrix} w_1(t) & w_2(t) \\ \dot{w}_1(t) & \dot{w}_2(t) \end{bmatrix}. \tag{13}$$

The stability of the trivial solution is determined by the Floquet multipliers which are the eigenvalues of the matrix $\mathbf{C}=\mathbf{W}^{-1}(\theta)\mathbf{W}(\theta+T)$ where θ is the initial time.

Now our task is reduced to finding the magnitudes of the Floquet multipliers. The two multipliers associated with (11) can be found easily by comparison with (4). Differentiating (11), we obtain a linear variational equation which is the same as that associated with small displacements δ from the orbitally stable limit cycle of the uncoupled oscillator when $\dot{h}$ is identified with δ. Identical variational equations yield identical Floquet multipliers, and Cesari [6] gives the Floquet multipliers for an orbitally stable limit cycle as $\lambda_1=1$, $\lambda_2<1$. The stability of the in-phase mode is therefore determined by equation (12). If the multipliers associated with (12) have magnitude less than one, the in-phase mode is stable.

Determining the multipliers associated with (12) is more difficult as we must find the matrix $\mathbf{C}$ by solving for two independent sets of initial conditions at $t=\theta$. For simplicity we choose initial conditions $\mathbf{W}(\theta)=\mathbf{I}$ so that $\mathbf{C}=\mathbf{W}^{-1}(\theta)\mathbf{W}(\theta+T)=\mathbf{W}(\theta+T)$.

It remains to find the solutions of (12) corresponding to these initial conditions. We cannot solve this equation exactly, since the periodic coefficient u^2 is not known exactly. We can, however, obtain an approximate solution for $v(t)$ using a perturbation scheme, the method of matched asymptotic expansions. This is the same method which Carrier [5] and Cole [8] used to find an approximation for the limit cycle, $u(t)$, of the uncoupled oscillator.

3. Limit cycle of the uncoupled oscillator. Here we present a brief summary of the results of Carrier [5] and Cole [8]. Figure 1 shows how the solution of the leading order perturbation equations in the four regions of distinct asymptotic behavior fit together to form one half-period of the limit cycle $u(t)$ of the uncoupled oscillator, eq. (4). Below we give the corresponding asymptotic expansions, the leading order perturbation equations, and the asymptotic behavior of the solution as the next region is approached.

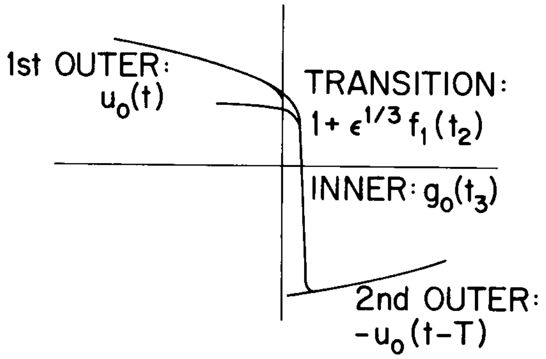

FIG. 1. *Matched asymptotic expansions approximation to the limit cycle.*

(14) 1st OUTER: $u = u_0(t) + \cdots,$

(15) $$\frac{du_0}{dt} = \frac{u_0}{1-u_0^2},$$

(16) $$u = 1 + \sqrt{-t} + \cdots, \qquad t \to 0-,$$

(17) TRANSITION: $u = 1 + \varepsilon^{1/3} f_1(t_2) + \cdots, \qquad t_2 = t/\varepsilon^{2/3},$

(18) $$\frac{d^2 f_1}{dt_2^2} + 2 f_1 \frac{df_1}{dt_2} + 1 = 0,$$

(19) $$u = 1 + \frac{\varepsilon}{t - \varepsilon^{2/3} t_0} + \cdots, \qquad t_2 \to t_0$$

where t_0 is the first zero of the Airy function $\mathrm{Ai}(-t)$,

(20) INNER: $u = g_0(t_3) + \cdots, \qquad t_3 = (t - \varepsilon^{2/3} t_0)/\varepsilon,$

(21) $$\frac{d^2 g_0}{dt_3^2} - (1 - g_0^2) \frac{dg_0}{dt_3} = 0,$$

(22) $$u = -2 + \cdots, \qquad t_3 \to +\infty,$$

(23) 2nd OUTER: $u = -u_0(t - T) + \cdots,$

where u_0 satisfies (15).

4. Solution of the variational equation. This section contains a summary of the solution of (12) by the method of matched asymptotic expansions. The expansions for v, the governing equations, and their solutions which satisfy the matching conditions are given below. The complete procedure used to derive these results is described in Appendix A. These expansions depend on those given in the preceding section for the limit cycle of the uncoupled oscillator. Thus, the same regions of distinct asymptotic behavior occur, along with an additional initial region near $t = \theta$ (see Fig. 2) which

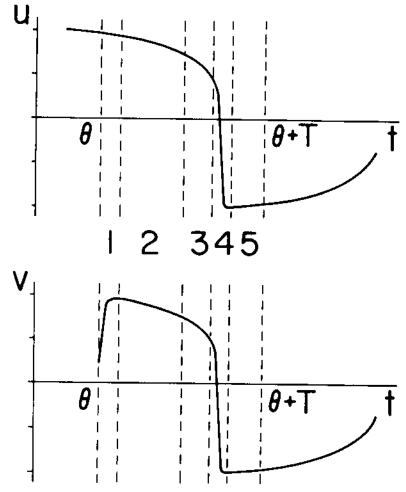

FIG. 2. *Regions associated with distinct asymptotic behavior of u and v.*

allows first-order matching with the initial conditions

(24) INITIAL: $v=y_0(t_1)+\varepsilon y_1(t_1)+\cdots, \quad t_1=(t-\theta)/\varepsilon,$

(25) $$\frac{d^2y_0}{dt_1^2}-(1-u_0(\theta)^2)\frac{dy_0}{dt_1}=0,$$

(26) $$v=v(\theta)+\cdots,$$

(27) 1st OUTER: $v=v_0(t)+\varepsilon v_1(t)+\cdots,$

(28) $$(1-u_0^2)\frac{dv_0}{dt}-pv_0=0,$$

(29) $$v=\frac{v(\theta)}{u_0(\theta)^p}u_0(t)^p,$$

(30) TRANSITION: $v=x_0(t_2)+\varepsilon^{1/3}x_1(t_2)+\cdots, \qquad t_2=t/\varepsilon^{2/3},$

(31) $$\frac{d^2x_0}{dt_2^2}+2f_1\frac{dx_0}{dt_2}=0,$$

(32) $$\frac{d^2x_1}{dt_2^2}+2f_1\frac{dx_1}{dt_2}=-(f_1^2+2f_2)\frac{dx_0}{dt_2}-px_0,$$

(33) $$v=\frac{v(\theta)}{u_0(\theta)^p}[1+\varepsilon^{1/3}pf_1(t_2)]+\cdots,$$

(34) INNER: $v=z_0(t_3)+\cdots, \qquad t_3=\dfrac{t-t_0\varepsilon^{2/3}}{\varepsilon},$

(35) $$\frac{d^2z_0}{dt_3^2}-(1-g_0^2)\frac{dz_0}{dt_3}=0,$$

(36) $$v=\frac{v(\theta)}{u_0(\theta)^p}[1-p+pg_0(t_3)]+\cdots,$$

(37) 2nd OUTER: $$v=\frac{(1-3p)v(\theta)}{u_0(\theta)^p}\left[\frac{u_0(t-T)}{2}\right]^p$$

(38) $$\Rightarrow v(\theta+T)=\frac{(1-3p)v(\theta)}{2^p}=-\frac{(1+3\alpha)}{4^\alpha}v(\theta).$$

5. Stability results for the in-phase mode. Now we are ready to employ the Floquet analysis referred to in § 2. Choose $\mathbf{W}(\theta)=\mathbf{I}$ so that $w_1(t)$ is the solution for $v(t)$ with initial conditions $v(\theta)=1$, $\dot{v}(\theta)=0$ and $w_2(t)$ is the solution for $v(t)$ with initial conditions $v(\theta)=0$, $\dot{v}(\theta)=1$. From (37) we see that the leading order approximation gives $w_2(t)\equiv 0$ and the Floquet matrix becomes

(39) $$\mathbf{C}=\begin{bmatrix} w_1(\theta+T) & 0 \\ \dot{w}_1(\theta+T) & 0 \end{bmatrix}+\cdots.$$

That is, the solution $w_2(t)$ vanishes to first order. The eigenvalues of $\mathbf{C}$ are then

(40) $$\lambda_1=w_1(\theta+T)+\cdots=-\frac{(1+3\alpha)}{4^\alpha}+\cdots, \qquad \lambda_2=0+\cdots.$$

Thus we see that λ_1 controls the stability of the in-phase mode. A plot of the leading term of λ_1 as a function of α and the corresponding stability results are shown in Fig. 3. Note that the in-phase mode is unstable for $0<\alpha<1$ as well as for $\alpha<-\frac{1}{2}$.

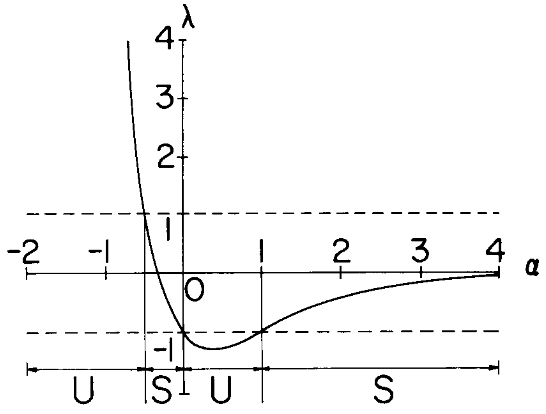

FIG. 3. *Plot of* $\lambda_1 = -(1+3\alpha)/4^{\alpha}$ *versus* α.

Due to our use of a perturbation method, these results are only expected to be valid for $\varepsilon \ll 1$. Can we extend the transition points $\alpha = -\frac{1}{2}$, 0, 1 into curves which define stability regions in the α, ε plane? Stability transitions occur when one of the multipliers has unit magnitude and this corresponds to the existence of a periodic solution of the variational equation (12). When $\alpha = -\frac{1}{2}$, (12) becomes

$$\varepsilon\ddot{v}-(1-u^2)\dot{v}=0 \tag{41}$$

which possesses the periodic solution $v=$ constant for any value of ε. Similarly, when $\alpha=0$, (12) becomes

$$\varepsilon\ddot{v}-(1-u^2)\dot{v}+v=0 \tag{42}$$

which possesses the periodic solution $v(t;\,\varepsilon)=u(t;\,\varepsilon)$ for all $\varepsilon>0$.

From Floquet theory, the presence of periodic solutions implies that the lines $\alpha=-\frac{1}{2}$, $\alpha=0$ are stability transition curves for the in-phase mode. The remaining curve containing the point $\alpha=1$, $\varepsilon=0$ cannot be easily determined analytically. Instead, we find this curve by using a fourth-order Runge-Kutta numerical integration scheme to obtain solutions of (4) and (12) for ε away from zero. These are then used in the Floquet analysis to numerically determine **C** and obtain its eigenvalues λ_1, λ_2.

The stability regions and transition curves in the α, ε plane are shown in Fig. 4. Note that the ε-axis has been compressed in a nonlinear fashion so that we can show $\varepsilon=\infty$ on a finite page. This is done because $\varepsilon=\infty$ is of special interest; it corresponds to the sinusoidal limit which we treated in our previous paper [18] using the two-variable expansion perturbation method. The results of that work are shown on the line $\varepsilon=\infty$ (note that some reinterpretation of those results necessary for $\alpha<0$ is provided in [17]) and the stability transition curves provide a consistent connection with our present perturbation results which appear on the line $\varepsilon=0$.

We can also link our results with those of Belair and Holmes [4] for two identical van der Pol oscillators with coupling in the velocities. To do this, we first obtain the variational equation corresponding to (12) for $\beta\neq 0$:

$$\varepsilon\ddot{v}-(1-2\beta-u^2)\dot{v}+(1+2\alpha)v=0. \tag{43}$$

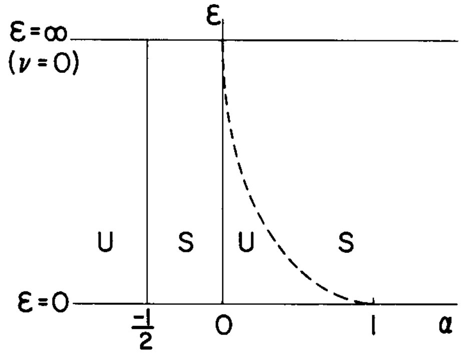

FIG. 4. *Regions of stability* (S) *and instability* (U) *for the in-phase mode.*

Note that (11) remains unchanged. We then employ the numerical Floquet analysis to obtain stability regions in the ξ, β plane for $\varepsilon = 0.01 \ll 1$. These results are shown in Fig. 5 and consistently connect our results for $\beta = 0$ with those of Belair and Holmes, who found that the in-phase mode is stable for $\alpha = 0$, $\beta > 0$ and unstable for $\alpha = 0$, $\beta < 0$.

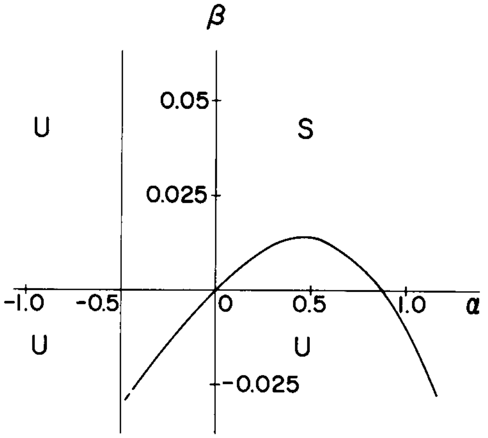

FIG. 5. *Numerically obtained stability results for the in-phase mode,* $\varepsilon = .01$. *Dashed portion represents uncertain continuation of curve due to limited computing precision.*

6. The out-of-phase mode. Let us now look for a periodic phase-locked motion where the two oscillators are one-half cycle out of phase, i.e.

$$x(t) = -y(t) \overset{\text{def}}{=} q(t). \tag{44}$$

Again concentrating on a pair of identical oscillators with displacement coupling, we see that $q(t)$ is the orbitally stable limit cycle of

$$\varepsilon \ddot{q} - (1 - q^2)\dot{q} + (1 + 2\alpha)q = 0. \tag{45}$$

We proceed, just as in the in-phase case, to find the homogeneous variational equations

$$\varepsilon \ddot{r} - (1 - q^2)\dot{r} + (1 + 2\alpha)r = 0, \tag{46}$$

$$\varepsilon \ddot{s} - (1 - q^2)\dot{s} + s = 0. \tag{47}$$

Let us now rescale time according to $\tau=(1+2\alpha)t$. Then (45)-(47) become

$$\varepsilon_1 q''-(1-q^2)q'+q=0, \tag{48}$$

$$\varepsilon_1 r''-(1-q^2)r'+r=0, \tag{49}$$

$$\varepsilon_1 s''-(1-q^2)s'+(1+2\alpha_1)s=0 \tag{50}$$

where $\varepsilon_1=(1+2\alpha)\varepsilon$, $\alpha_1=-\alpha/(1+2\alpha)$ and primes denote differentiation with respect to τ. Note that upon identifying ε_1, α_1, τ, r, s with ε, α, t, h, v (48), (49), and (50) become exactly (4), (11), and (12) which we have analysed in the previous sections. Thus we obtain the stability of the out-of-phase mode for the system with parameters ε, α directly from the stability of the in-phase mode with parameters ε_1, α_1 and the results are shown in Fig. 6. Note that for $\alpha<-\frac{1}{2}$, the only equilibrium point in the q, $\dot{q}$ phase plane becomes a saddle point and the Poincaré index theorem then implies that no periodic orbits can exist. Therefore, in this region the out-of-phase mode does not exist.

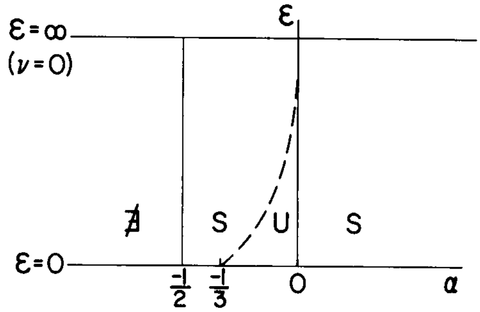

FIG. 6. *Stability of the out-of-phase mode. S = stable, U = unstable, ∄ = does not exist.*

Note also that the results on the line $\varepsilon=\infty$ once again coincide with our earlier results for the sinusoidal limit [18]. See [17] for a more complete discussion of existence and stability of the out-of-phase mode.

7. Summary and conclusions. Using Floquet theory and the method of matched asymptotic expansions, we have determined the stability of the in-phase and out-of-phase modes of a pair of identical van der Pol oscillators with linear diffusive coupling in the displacements for the small ε (relaxation) limit. Using numerical integration, these results have been extended to the entire ε, α half-plane ($\varepsilon>0$) and feature three stability transition curves which consistently connect the behavior in the sinusoidal and relaxation limits. Numerical methods were also used to determine the stability of the in-phase mode in a central region of the α, β plane for $\varepsilon \ll 1$. Our findings are in agreement with the previously existing results of [4] and [18].

The central role of Floquet theory in our analysis should be emphasized. Neither the method of matched asymptotic expansions nor numerical integration can be expected to yield solutions of the variational equations which are valid as $t\to\infty$, although this is of essential importance for stability problems. This difficulty is resolved by Floquet theory which enables us to determine stability as $t\to\infty$, given the behavior of the system over one period.

It is interesting to note that for this problem the large ε limit (i.e. the case of nearly sinusoidal oscillations) gives a very misleading picture. As $\varepsilon\to\infty$, we see from Figs. 4, 6 that both the in-phase and out-of-phase modes are stable for all $\alpha>-\frac{1}{2}$. The asymptotic and numerical analyses presented in this paper, however, show that a region of instability for each of these modes grows out of the $\varepsilon=\infty$ line (cf. Figs. 4, 6).

Appendix A. Derivation of the solution of the variational equation. Here we employ the results of § 3 to obtain an approximate solution to the variational equation (12).

Let us begin with a straightforward perturbation series approach. We look for an outer solution of the form

$$v(t;\varepsilon)=v_0(t)+\varepsilon v_1(t)+\cdots \tag{A1}$$

and substitute into (12). Collecting terms of leading order in ε gives

$$(1-u_0^2)\dot{v}_0+pv_0=0 \tag{A2}$$

which we solve using (15) to obtain v_0.

$$\frac{pv_0}{1-u_0^2}=\frac{dv_0}{dt}=\frac{dv_0}{du_0}\frac{du_0}{dt}=\frac{dv_0}{du_0}\frac{u_0}{1-u_0^2}\Rightarrow\frac{dv_0}{v_0}=p\frac{du_0}{u_0}. \tag{A3}$$

Integration yields

$$v_0=c_3u_0^p \tag{A4}$$

which is valid in the first outer region (#2 in Fig. 2). However, this solution contains only one arbitrary constant and cannot in general satisfy given initial conditions $v(\theta)$, $\dot{v}(\theta)$.

This indicates the presence of an initial boundary layer at $t=\theta$ to link the outer solution to the initial conditions. Here we look for a solution of the form

$$v(t;\varepsilon)=y_0(t_1)+\varepsilon y_1(t_1)+\cdots,\qquad t_1=(t-\theta)/\varepsilon. \tag{A5}$$

Substituting into (12), expanding $u_i(\varepsilon t_1)$ in a Taylor series

$$u_i(\varepsilon t_1)=u_i(\theta)+\varepsilon t_1\dot{u}_i(\theta)+\cdots, \tag{A6}$$

and collecting terms of leading order in ε gives

$$\frac{d^2y_0}{dt_1^2}-(1-u_0(\theta)^2)\frac{dy_0}{dt_1}=0. \tag{A7}$$

To simplify notation, let $A=1-u_0(\theta)^2$. Then (A7) has the solution

$$y_0=c_1+c_2\,e^{At_1}. \tag{A8}$$

Note that, if $A=1-u_0(\theta)^2>0$, this solution blows up as $t_1\to\infty$ and matching with a bounded outer solution would be impossible. Thus, for this expansion to be valid, the initial region must lie within an outer region of (4) (where $u_0(\theta)>1$) and away from the transition and inner regions (as shown in Fig. 2). If this is the case, we can satisfy the initial conditions to first order by matching leading terms as follows

$$\begin{aligned} &v(\theta)=c_1+c_2+\cdots, &&\frac{dv(\theta)}{dt}=\frac{A}{\varepsilon}c_2+\cdots,\\ &\Rightarrow c_1=v(\theta), &&c_2=0. \end{aligned} \tag{A9}$$

Matching with the first outer solution (A4) then requires

$$c_1=c_3u_0(\theta)^p. \tag{A10}$$

Next we treat the transition region (region 3 in Fig. 2) where we seek a solution of the form

(A11) $$v(t;\,\varepsilon)=x_0(t_2)+\varepsilon^{1/3}x_1(t_2)+\cdots,\qquad t_2=\frac{t}{\varepsilon^{2/3}}.$$

Substituting (A11) and (22) into (12) and collecting terms of like powers of ε gives

(A12) $$\frac{d^2x_0}{dt_2^2}+2f_1\frac{dx_0}{dt_2}=0,$$

(A13) $$\frac{d^2x_1}{dt_2^2}+2f_1\frac{dx_1}{dt_2}=-(f_1^2+2f_2)\frac{dx_0}{dt_2}-px_0.$$

Integrating (A12) twice and setting $f_1=d(\ln\varphi)/dt_2$ yields

(A14) $$x_0=k\int\varphi^{-2}\,dt_2+c_4.$$

To match with the first outer solution, we must determine the asymptotic behavior of x_0 as $t_2\to-\infty$. Using (17) and (18) we find that $\int\varphi^{-2}\,dt_2\to\infty$ as $t_2\to-\infty$ (see Storti [17, p. 53]). Thus to achieve a match with the first outer solution we require $k=0$ and (A14) simplifies to $x_0=c_4=\text{constant}$. To discover the nature of the time variation of the transition solution, we must find the next term in the expansion. Since $x_0=\text{constant}=c_4$, (A13) reduces to

(A15) $$\frac{d^2x_1}{dt_2^2}+2f_1\frac{dx_1}{dt_2}+pc_4=0$$

which can be solved directly from our previous solutions. From (A12), we obtain the complementary solution

(A16) $$x_{1c}=k\int\varphi^{-2}\,dt_2+c_5$$

where k must once again vanish to eliminate a singularity which would prevent matching. Comparison with (18) gives the particular solution $x_{1p}=pc_4f_1$, and the transition expansion becomes

(A17) $$v(t;\,\varepsilon)=c_4+\varepsilon^{1/3}(pc_4f_1+c_5)+\cdots.$$

The asymptotic behavior as $t_2\to-\infty$, using (16), is given by

(A18) $$v(t;\,\varepsilon)=c_4(1+p\sqrt{-t})+\varepsilon^{1/3}c_5+\cdots.$$

This is to be matched to the outer solution $v_0=c_3u_0^p$ whose behavior as $t\to0^-$ is (from (16))

(A19) $$v(t;\,\varepsilon)=c_3(1+\sqrt{-t}+\cdots)^p+O(\varepsilon)=c_3(1+p\sqrt{-t}+\cdots)+O(\varepsilon).$$

Matching these expressions requires

(A20) $$c_4=c_3,\qquad c_5=0.$$

Now we move on to the inner region (#4 in Fig. 2) and look for a solution

(A21) $$v(t;\,\varepsilon)=z_0(t_3)+\cdots,\qquad t_3=\frac{t-t_0\varepsilon^{2/3}}{\varepsilon}.$$

Substituting (A21) and (20) into (12) and collecting leading powers of ε gives

(A22) $$\frac{d^2 z_0}{dt_3^2}-(1-g_0^2)\frac{dz_0}{dt_3}=0.$$

By inspection, we find the solution $z_0=c_6=\text{constant}$ and comparison with (21) yields a second independent solution $z_0=g_0$. The general solution is then

(A23) $$z_0=c_6+c_7 g_0$$

which is to be matched with the transition solution as $t_3\to-\infty$, $g_0\to 1$. Hence, using (19) we find

(A24) $$v(t;\,\varepsilon)=z_0+\cdots=c_6+c_7\left(1+\frac{\varepsilon}{t-\varepsilon^{2/3}t_0}\right)+\cdots$$

which must agree with the expression arising from the transition expansion (see (A18))

(A25) $$v(t;\,\varepsilon)=c_4\left(1+\frac{\varepsilon}{t-\varepsilon^{2/3}t_0}\right)^p+\cdots=c_4\left(1+\frac{\varepsilon p}{t-\varepsilon^{2/3}t_0}\right)+\cdots.$$

Matching requires

(A26) $$c_6+c_7=c_4,\qquad c_7=pc_4.$$

To complete the solution, we move on to the second outer region (#5 in Fig. 2). Here a solution similar to that found in the first outer region (#2) is once again valid, but now we are on the negative branch of $u(t;\,\varepsilon)$ which is obtained from the positive branch by multiplying by -1 and translating by the period T. To avoid inconveniences associated with raising negative numbers to fractional powers, we write the solution as

(A27) $$v(t;\,\varepsilon)=c_8\left[\frac{-u_0(t-T)}{-2}\right]^p=c_8\left[\frac{u_0(t-T)}{2}\right]^p$$

and the behavior near the inner region is

(A28) $$-u_0(t-T)=-2+\cdots,\qquad v(t;\,\varepsilon)=c_8+\cdots.$$

This must agree with the inner solution as $t_3\to\infty$

(A29) $$v(t;\,\varepsilon)=c_6+c_7 g_0+\cdots=c_6-2c_7+\cdots$$

which implies

(A30) $$c_8=c_6-2c_7.$$

Combining the matching results of (A9), (A10), (A20), (A26), and (A30), we find

(A31) $$c_8=\frac{(1-3p)v(\theta)}{u_0(\theta)^p}.$$

Thus (A27) yields a first order approximation for $v(\theta+T)$ arising from initial conditions $v(\theta)$, $\dot v(\theta)$:

(A32) $$v(\theta+T)=(1-3p)\frac{v(\theta)}{u_0(\theta)^p}\left[\frac{u_0(t-T)}{2}\right]^p\Bigg|_{t=\theta+T}=\frac{(1-3p)v(\theta)}{2^p}=-\frac{(1+3\alpha)}{4^\alpha}v(\theta).$$

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