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A NUMERICAL INVESTIGATION OF THE DYNAMICS OF A SYSTEM OF TWO TIME-DELAY COUPLED RELAXATION OSCILLATORS

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Abstract. In this paper we examine the dynamics of two time-delay coupled relaxation oscillators of the van der Pol type. By integrating the governing differential-delay equations numerically, we find the various phase-locked motions including the in-phase and out-of-phase modes. Our computations reveal that depending on the strength of coupling (α) and the amount of time-delay (τ), the in-phase (out-of-phase) mode may be stable or unstable. There are also values of α and τ for which the in-phase and out-of-phase modes are both stable leading to birhythmicity. The results are illustrated in the α-τ parameter plane. Near the boundaries between stability and instability of the in-phase (out-of-phase) mode, many other types of phase-locked motions can occur. Several examples of these phase-locked states are presented.

1. **Introduction.** A relaxation oscillator is characterized by a periodic motion which consists of a fast regime and a slow regime within each period. Coupled relaxation oscillators are often used to model the dynamics of a variety of physical, chemical and biological systems. There is a vast literature on this subject (see, for example, [6], [2], [7], [11]). In many applications the coupling process necessarily involves a time-delay. Delays occur due to finite rates of transport such as diffusion and signal transmission. Despite this fact, effects of time-delay have not been considered in these earlier works on coupled relaxation oscillators. Currently there is a great deal of interest in the study of relaxation oscillators with time-delay coupling. These studies deal with systems of neuronal oscillators that are coupled by a time-delay through their synapses ([4], [1], [14]). Effects of discrete and distributed time-delays in many other systems have also been investigated recently (see, for example, [5], [8]).

In this paper we investigate the dynamics of a pair of relaxation oscillators of the van der Pol type with time-delay coupling. The motion of this system is governed by the following differential-delay equations:

$$\begin{aligned} \epsilon\ddot{x}_1 - (1-x_1^2) + x_1 &= \alpha[x_2(t-\tau) - x_1(t-\tau)] \\ \epsilon\ddot{x}_2 - (1-x_2^2) + x_2 &= \alpha[x_1(t-\tau) - x_2(t-\tau)] \end{aligned} \tag{1.1}$$

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Here x_1 and x_2 are the dependent variables (e.g., voltages), t is time, and a dot over a variable denotes a time-derivative. The coupling strength and time-delay are designated by the symbols α and τ, respectively, and the parameter ϵ is assumed to be much less than unity.

If time-delays are introduced into the electronic circuit considered by Saito [7], the behavior of the system of time-delay coupled relaxation oscillators can be modeled by equations such as (1.1) above. Using discrete time-delay elements in Saito's circuit, we have performed laboratory experiments to study the effects of the strength and amount of delay on the dynamic behavior of the coupled relaxation oscillators. These experimental results will be reported in a future publication. In this paper we present the results of a numerical investigation of the system of equations (1.1) which exhibits the nonlinear dynamics of such circuits.

In an earlier work, Storti and Rand [9] examined the system of equations (1.1) in the absence of time-delay (i.e., $\tau = 0$). Using the method of matched asymptotic expansions and Floquet theory, they determined the conditions for stability of the in-phase and out-of-phase modes. They showed that in the limit $\epsilon \to 0$, the in-phase mode is stable for $-1/2 < \alpha < 0$ and $\alpha > 1$, and is unstable for $\alpha < -1/2$ and $0 < \alpha < 1$. The out-of-phase mode, on the other hand, is found to be stable when $-1/2 < \alpha < -1/3$ and $\alpha > 0$; it is unstable for $-1/3 < \alpha < 0$, whereas for $\alpha < -1/2$, the out-of-phase mode does not exist. This work was extended recently by Storti and Reinhall [10] to include the effect of velocity coupling.

Grasman [2] considered the time-delayed system (1.1) and reformulated the equations in terms of the Lienard variables as follows:

$$\begin{aligned} \epsilon\dot{x}_1 &= y_1 + x_1 - \frac{x_1^3}{3}, \qquad \dot{y}_1 = -x_1 + \alpha[x_2(t-\tau) - x_1(t-\tau)] \\ \epsilon\dot{x}_2 &= y_2 + x_2 - \frac{x_2^3}{3}, \qquad \dot{y}_2 = -x_2 + \alpha[x_1(t-\tau) - x_2(t-\tau)] \end{aligned} \tag{1.2}$$

He analyzed these equations for weak coupling by assuming that $0 < \epsilon \ll \alpha < 1$. Under this condition the coupling does not perturb the limit cycle of the individual oscillators; it only alters their phases. Grasman [2] derived solutions for the phase functions in order to describe the dynamics of the coupled system.

In this study we examine equations (1.1) for a range of values of the coupling strength and time-delay. The objective is to determine the possible phase-locked motions that this system can exhibit and find the values of α and τ for which such motions can occur. Our approach is based on a numerical solution of these equations. The computational procedure we have used is discussed in detail in [3].

Equations (1.1) constitute a system of nonlinear differential equations with time-delay. An analytical treatment of these equations for *all* values of ϵ is impossible. In the relaxation limit $\epsilon \to 0$ considered here, these equations present a singular perturbation problem. We have tried to solve these equations analytically using a singular perturbation technique before attempting a purely numerical simulation. Unfortunately, however, the equations that result from the application of a singular perturbation technique are not amenable to analytical solutions. This is why a fully numerical approach was adopted for the solution of these equations.

The results of this study may be summarized as follows. The time-delayed system given by equations (1.1) or (1.2) undergoes a variety of phase-locked motions including an in-phase mode and an out-of-phase mode. Depending on the coupling strength and the amount of time-delay, the in-phase (out-of-phase) mode may be stable or unstable. We also find that for certain values of these parameters the in-phase and out-of-phase modes are both stable leading to birhythmicity. The results are illustrated in the α-τ parameter plane. Near the boundaries of stability and instability of the in-phase (out-of-phase) mode, many other types of phase-locked states may exist. Several examples of these phase-locked states are presented.

2. **The In-Phase and Out-of-Phase Modes: Existence and Stability.** It is easy to see that for all values of α and τ , equations (1.1) and (1.2) admit the solution: $x_1 = x_2 = u(t)$, representing the in-phase mode. Here $u(t)$ describes the relaxation limit cycle of the uncoupled van der Pol oscillator which is governed by the equation

$$\epsilon\ddot{u} - (1-u^2)\dot{u} + u = 0 \tag{2.1}$$

Equation (2.1) has been studied in great detail by many researchers. Here we simply note that for $\epsilon = 0.05$, which is the value used in all our computations, the period of the limit cycle in (2.1) is approximately equal to 2.43 (see, for example, [2]).

Equations (1.1) and (1.2) also admit the out-of-phase solution: $x_1 = -x_2 = v(t)$, where $v(t)$ satisfies the following equation:

$$\epsilon\ddot{v} - (1-v^2)\dot{v} + v = -2\alpha v(t-\tau) \tag{2.2}$$

For given values of α and τ, this equation determines the existence of the out-of-phase mode. In order to investigate the stability of the in-phase and out-of-phase modes we integrate equation (1.2) numerically. Consider first the in-phase mode. To find the values of α and τ for which the in-phase mode is stable, we proceed as follows. Choose a value of $\epsilon \ll 1$; as mentioned above we have used $\epsilon = 0.05$. Now choose the parameters α and τ, and select an initial condition which is close to the in-phase mode: $x_1 = x_2$. We have used $x_1 = 1.3$, $x_2 = 1.35$, with $y_1 = y_2 = 0$. Next we integrate the uncoupled equations in (1.2), i.e., with $\alpha = 0$, backward from $t = 0$ to $t = -\tau$. The solution thus obtained is subsequently used as the set of initial conditions over the interval $[-\tau, 0]$ to integrate the coupled time-delay equations in (1.2). During each step of this integration we must take into account the values of (x_1, x_2, y_1, y_2) at a time τ units earlier. We have adopted a fourth order Runge-Kutta method for our integration. The details of this technique are given in [3]. The stability of the in-phase mode is established by observing if the computed trajectories in the four dimensional space approach the in-phase mode with time.

To investigate the stability of the out-of-phase mode, we choose the parameters α and τ, and use an initial condition close to this mode. In our simulation we have used $x_1 = 1.3$, $x_2 = -1.35$, and $y_1 = y_2 = 0$. Next we integrate the uncoupled equations in (1.2) from $t = 0$ to $t = -\tau$, analogous to the situation with the in-phase mode. Using this solution as the set of initial conditions over the interval $[-\tau, 0]$, a numerical solution of the differential-delay equations (1.2) is found. The above calculations are repeated for several values of α and τ. The results are depicted in Figures 1 and 8 for the in-phase and out-of-phase modes, respectively.

3. **Results and Discussion.** Figure 1 reveals that for a given value of α with $\alpha > 0.68$, the in-phase mode remains stable for small delays, but it loses stability when the delay exceeds a critical value given by the transition curve. For α lying in the interval $-0.5 < \alpha < 0$, the in-phase mode remains stable for much longer delays beyond which it becomes unstable. There are also several thin regions around the

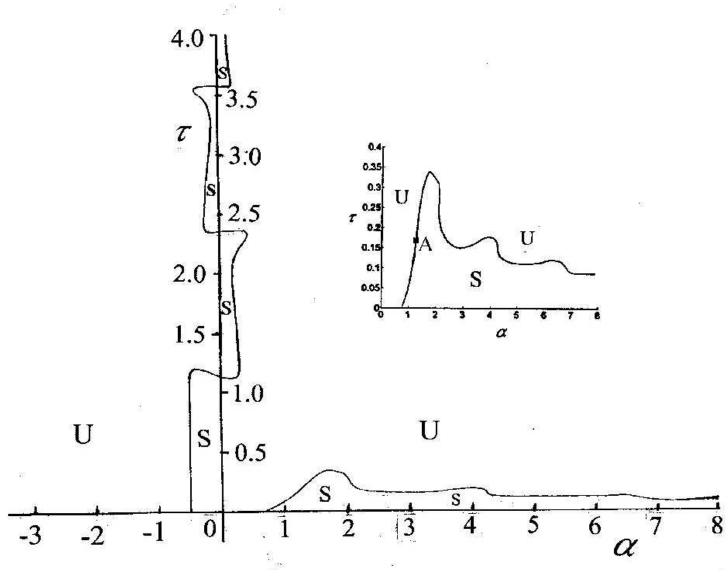

FIGURE 1. Stability results for the in-phase mode in the α-τ parameter plane for $\epsilon = 0.05$. S = Stable, U = Unstable.

τ-axis (i.e., for small α) within which the in-phase mode is stable. These regions alternate on the positive and negative sides of the τ-axis with each region spanning approximately half the period of the uncoupled oscillators. The in-phase mode is unstable in the remainder of the α-τ plane. The inset in Figure 1 shows the transition curve over the interval $\alpha > 0.68$ in more detail. Near the boundaries of stability and instability of the in-phase mode, many other types of phase-locked states may exist. Consider, for instance, the values of α and τ in the vicinity of the point A on the transition curve (see the inset in Figure 1). In particular, consider $\tau = 0.16$ and several values of α, namely, $\alpha = 1.15$, 1.2,1.205 and 1.208. Figure 2 depicts the phase-locked modes that exist at these parameter values. Note that since equations (1.1) are invariant under the transformation which interchanges x_1 and x_2, each of the periodic motions which is displayed in Figures 2-7 and Figures 9-11, is accompanied by a periodic motion which is its reflection in the line $x_1 = x_2$. Next consider $\alpha = 1.208$ and $\tau = 0.16$, 0.162, 0.165 and 0.17. The various phase-locked states for these parameters are illustrated in Figure 3. We now fix $\tau = 0.16$ and see if there are other values of α at which phase-locked modes different from the in-phase and out-of-phase modes can exist. We find that for α lying between 2.5 and 3.3, many such phase-locked motions can occur. This is the interval with $\tau = 0.16$ over which the transition curve has an upward concavity. Figure 4 portrays the phase-locked mode corresponding to $\alpha = 3.0$ and $\tau = 0.16$. Consider another interval in the α-τ plane in which the transition curve is concave upward. In particular, consider the horizontal line given by $\tau = 0.115$ with values of α between 4.8 and 5.7. For every point on this line there exists a phase-locked state which is different from the in-phase and the out-of-phase modes. The phase-locked state corresponding to $\alpha = 5.0$ and $\tau = 0.115$ is displayed in Figure 5. Another interval with upward concavity lies over $\tau = 0.085$ and $7.1 < \alpha < 7.9$. Figure 6 shows the phase-locked motion at the point given by $\alpha = 7.7$ and $\tau = 0.085$ lying in this interval. As a final example, consider the point with parameter values $\alpha = -0.5$ and $\tau = 0.75$, which lies near the boundary between stability and instability on the negative side of the α-axis. The phase-locked mode for this case is shown in Figure 7. A variety of other phase-locked motions can occur near the boundaries of stability and instability of the in-phase mode. For the sake of brevity we will not discuss them further.

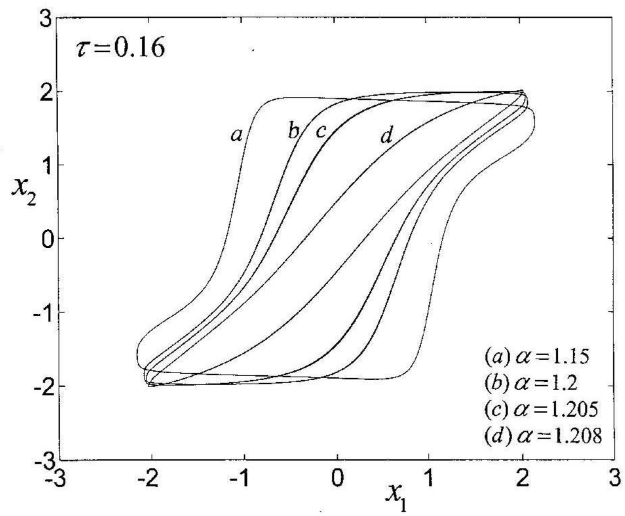

FIGURE 2. Phase-locked states near the point A on the inset of Figure 1. These states correspond to $\tau = 0.16$ and (a) $\alpha = 1.15$, (b) $\alpha = 1.2$, (c) $\alpha = 1.205$ and (d) $\alpha = 1.208$, respectively.

The numerical results for the stability of the out-of-phase mode are displayed in Figure 8. This figure reveals that for positive coupling strength ($\alpha > 0$), the out-of-phase mode is stable for all values of τ except in thin regions adjacent to the τ-axis where it is unstable. When the coupling strength is negative ($\alpha < 0$), there are also thin regions adjacent to the τ-axis where the out-of-phase mode is unstable. Lying to the left of these thin regions and to the right of the line $\alpha = -0.5$, there is a stable region for the out-of-phase mode. At $\alpha = -0.5$, the out-of-phase mode undergoes an infinite period bifurcation in which $x_1(t) = -x_2(t)$ remains constant with time. This constant solution can also be readily observed from equation (2.2) with $\alpha = -0.5$. For α less than this value, the out-of-phase mode ceases to exist. Analogous to the situation in Figure 1, other phase-locked modes are found near the boundaries of stability and instability of the out-of-phase mode. Figure 9 shows the phase-locked mode corresponding to $\alpha = 0.05$ and $\tau = 1.25$. Another phase-locked mode which occurs for $\alpha = -0.025$ and $\tau = 2.7$ is shown in Figure 10.

By superimposing Figures 1 and 8, we see that there are several overlapping regions in the α-τ parameter plane where the in-phase and out-of-phase modes are both stable. These regions of birhythmicity are illustrated in Figure 11. Around the τ-axis, Yeung and Strogatz [13] found similar overlapping regions between the

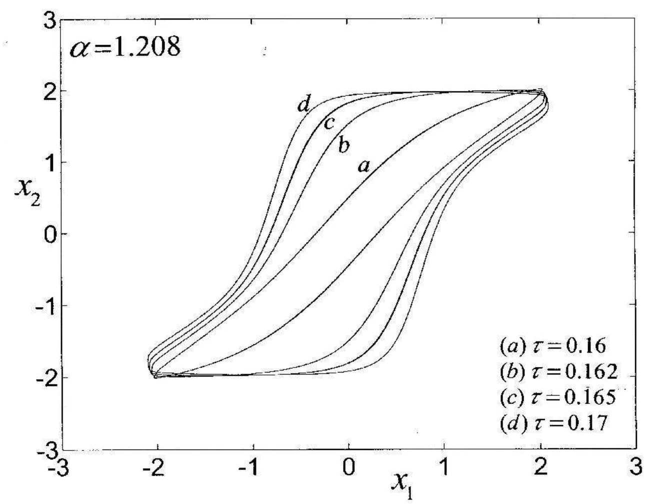

FIGURE 3. Phase-locked states near the point A on the inset of Figure 1. These states correspond to $\alpha = 1.208$ and (a) $\tau = 0.16$, (b) $\tau = 0.162$, (c) $\tau = 0.165$ and (d) $\tau = 0.17$, respectively.

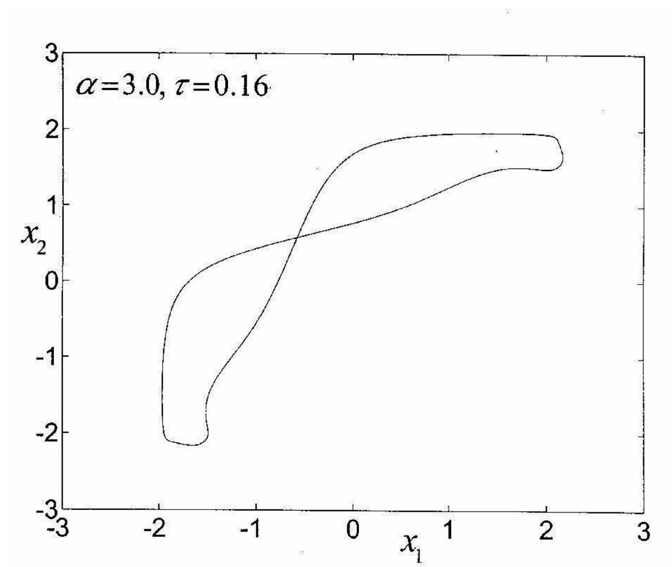

FIGURE 4. Phase-locked state for $\alpha = 3$ and $\tau = 0.16$ in Figure 1.

synchronized and incoherent states in their study of the Kuramoto model of oscillators with time-delay coupling. Wirkus and Rand [12] found parameter regions in which both the in-phase and out-of-phase modes are stable in a study of two van der Pol oscillators with time-delay (in the limit of non-relaxation behavior).

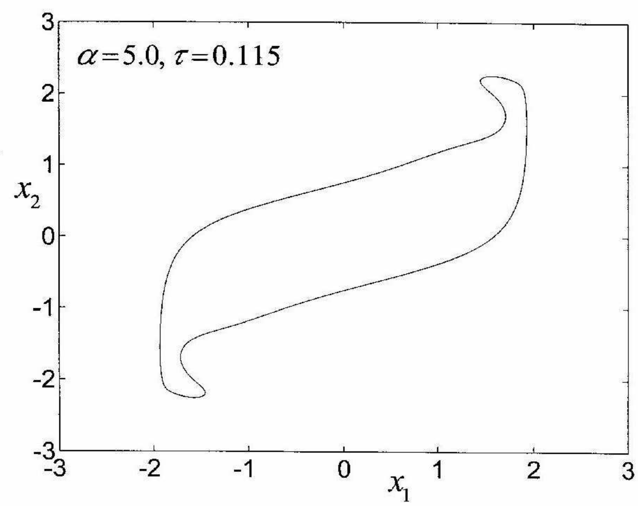

FIGURE 5. Phase-locked state for $\alpha = 5$ and $\tau = 0.115$ in Figure 1.

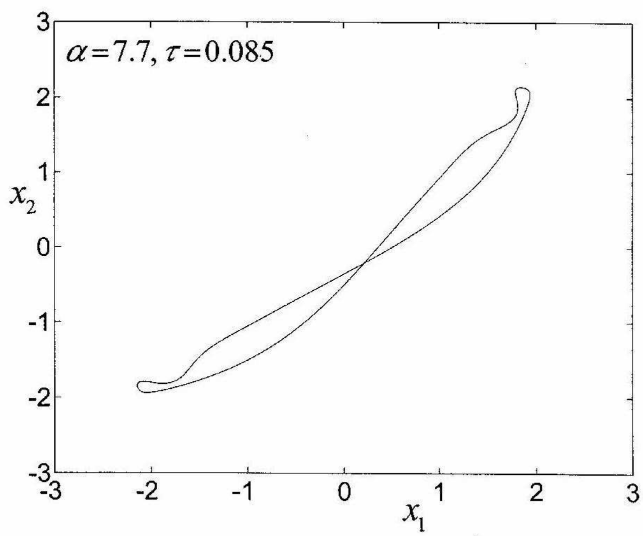

FIGURE 6. Phase-locked state for $\alpha = 7.7$ and $\tau = 0.085$ in Figure 1.

4. **Concluding Remarks.** We have examined the dynamics of two identical relaxation oscillators of the van der Pol type which are coupled by terms with time-delay. By integrating the governing differential-delay equations numerically, we have found various phase-locked motions. The results are presented in the α-τ parameter plane. Our computations reveal that depending on the strength of coupling (α) and the amount of time-delay (τ), a stable in-phase mode or a stable out-of-phase mode

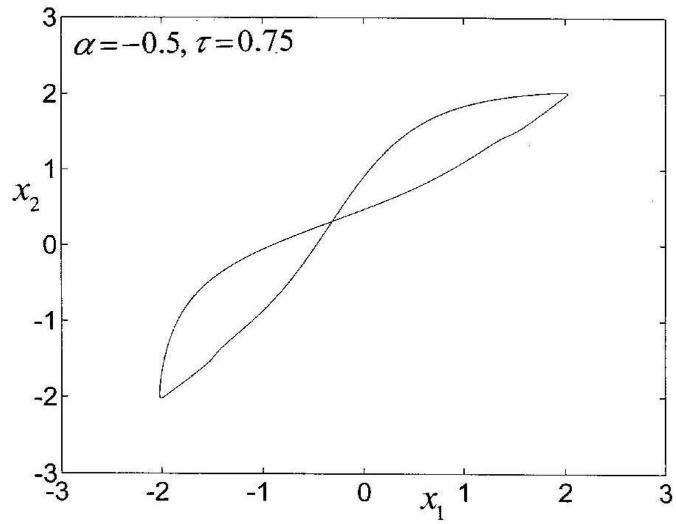

FIGURE 7. Phase-locked state for $\alpha = -0.5$ and $\tau = 0.75$ in Figure 1.

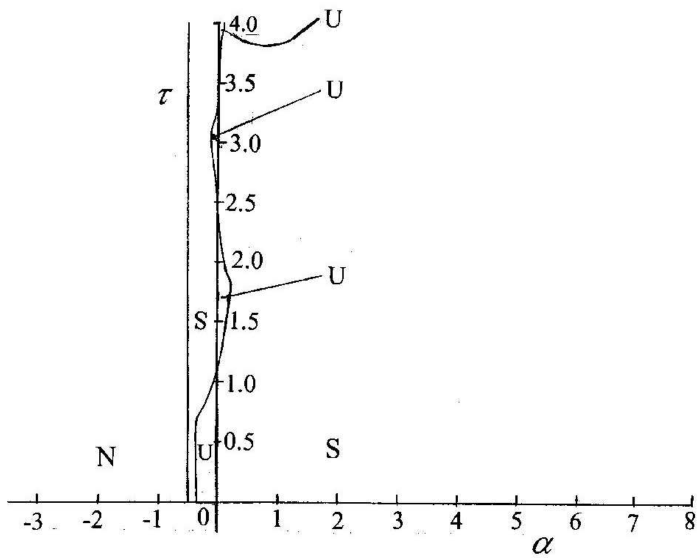

FIGURE 8. Stability results for the out-of-phase mode in the α-τ parameter plane for $\epsilon = 0.05$. S = Stable, U = Unstable, N = Does not exist.

may exist. For other values of these parameters the in-phase or the out-of-phase mode may become unstable. We also found that near the boundaries of regions

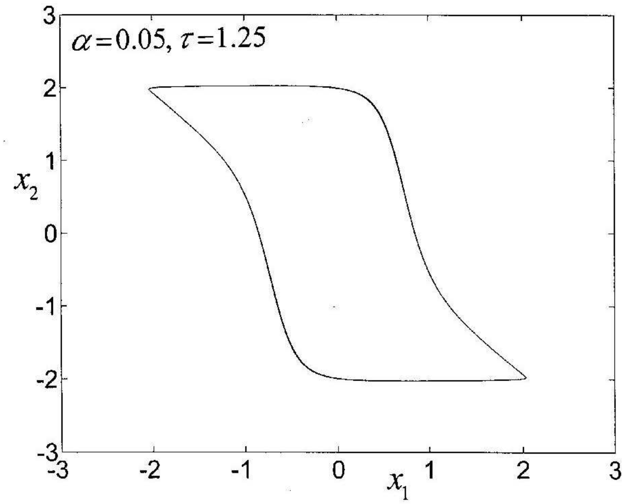

FIGURE 9. Phase-locked state for $\alpha = 0.05$ and $\tau = 1.25$ in Figure 8.

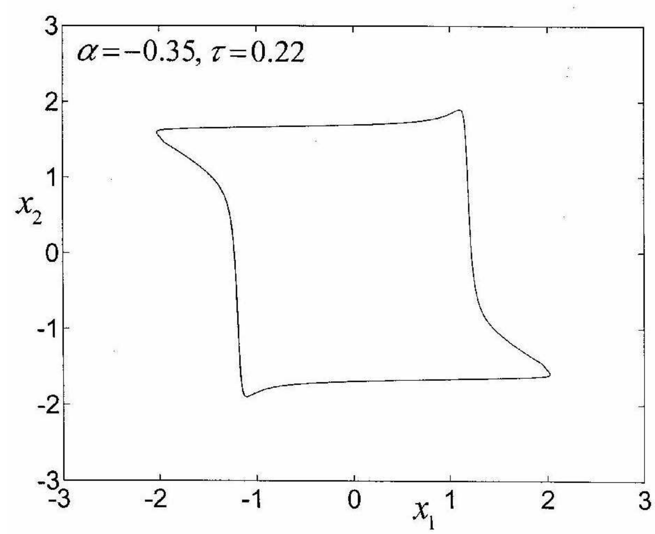

FIGURE 10. Phase-locked state for $\alpha = -0.35$ and $\tau = 0.22$ in Figure 8.

of stability and instability of these modes in the α-τ parameter plane, many other phase-locked motions can occur. Several examples of these phase-locked states have been presented.

In the above development we have used a value of $\epsilon = 0.05$. We have performed numerical simulations for other small values of ϵ as well, and found that the results

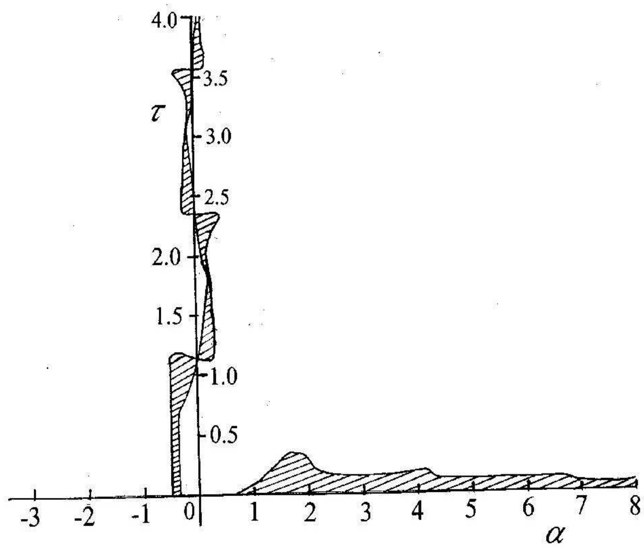

FIGURE 11. Regions of birhythmicity in the α-τ parameter plane for $\epsilon = 0.05$. These regions are shown shaded in the figure.

do not change appreciably and no new phenomena appear. In particular, the bifurcation curves etc. remain qualitatively the same; they only move slightly depending on the specific value of ϵ used.

REFERENCES

[1] S.R. Campbell and D.L. Wang, *Relaxation oscillators with time-delay coupling*, Physica D, 111 (1998), 151–178.

[2] J. Grasman, Asymptotic Methods for Relaxation Oscillations and Applications, Springer-Verlag, New York, 1987.

[3] E. Hairer, G. Norsett, and G. Warner, Solving Ordinary Differential Equations I, Springer-Verlag, New York, 1987.

[4] S.G. Lee, S. Kim, and H. Kook, *Synchrony and clustering in two and three synaptically coupled Hodgkin-Huxley neurons with a time delay*, Int. J. Bifurcation and Chaos, 7 (1997), 889–895.

[5] J. Li, Y. Kuang, and B. Li, *Analysis of IVGTT glucose-insulin interaction models with time delay*, Discrete and Continuous Dynamical Systems - Series B, 1 (2001), 103–124.

[6] K. Radparvar and B.Z. Kaplan, *Experimental and analytical investigations of synchronization dynamics of two coupled multivibrators*, IEEE Trans. Circuits Syst., 32 (1985), 1072– 1078.

[7] T. Saito, *On a coupled relaxation oscillator*, IEEE Trans. Circuits Syst., 35 (1988), 1147–1155.

[8] D. Schley, and S.A. Gourley, *Linear and nonlinear stability in a differential ecotoxicological model with time delays*, Discrete and Continuous Dynamical Systems - Series B, 2 (2002), 575–590.

[9] D. Storti, and R. Rand, *Dynamics of two strongly coupled relaxation oscillators*, SIAM J. Appl. Math., 46 (1986), 56–67.

[10] D. Storti, and R. Reinhall, *Stability of in-phase and out-of-phase modes for a pair of linearly coupled van der Pol oscillators*, in "Nonlinear Dynamics: The Richard Rand 50th Anniversary Volume", ed. A. Guran, pp. 1-23, World Scientific, Singapore, 1997.

[11] E.I. Volkov, and M.N. Stolyarov, *Birhythmicity in a system of two coupled oscillators*, Phys. Lett A, 159 (1991), 61–66.

[12] S. Wirkus, and R. Rand, *Dynamics of two coupled van der Pol oscillators with delay coupling*, Nonlinear Dynamics, 30 (2001), 205–221.
[13] M.K.S. Yeung, and S.H. Strogatz, *Time delay in Kuramoto model of coupled oscillators*, Phys. Rev. Lett., 82 (1999), 648–651.
[14] T. Yoshinga, Y. Sano and H. Kawakami, *A method to calculate bifurcations in synaptically coupled Hodgkin-Huxley equations*, Int. J. Bifurcation and Chaos, 9 (1999), 1451–1458.

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