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Parametric Excitation and Evolutionary Dynamics

Parametric excitation refers to dynamics problems in which the forcing function enters into the governing differential equation as a variable coefficient. Evolutionary dynamics refers to a mathematical model of natural selection (the "replicator" equation) which involves a combination of game theory and differential equations. In this paper we apply perturbation theory to investigate parametric resonance in a replicator equation having periodic coefficients. In particular, we study evolution in the Rock-Paper-Scissors game, which has biological and social applications. Here periodic coefficients could represent seasonal variation. We show that 2:1 subharmonic resonance can destabilize the usual "Rock-Paper-Scissors" equilibrium for parameters located in a resonant tongue in parameter space. However, we also show that the tongue may be absent or very small if the forcing parameters are chosen appropriately. [DOI: 10.1115/1.4023473]

1 Introduction

Evolutionary dynamics formalizes the process of evolution by combining game theory with differential equations [1–3]. Evolution is driven by natural selection: organisms with greater fitness (i.e., number of offspring) tend to become more common, while less fit organisms are driven to extinction. To describe evolution mathematically, game theory is used to represent the fitness of each type of organism in a given population. Then, differential equations describe how the abundances of each type of organism change based on those fitnesses.

Organism types can be thought of as 'strategies' in a game theoretic sense, which interact and earn payoffs (representing reproductive success) based on the strategy of each interacting agent. Here we consider one of the canonical games from evolutionary game theory, "Rock-Paper-Scissors" (RPS). There are three possible strategies: rock (R), paper (P) and scissors (S). As in the children's game of the same name, rock beats scissors, scissors beats paper, and paper beats rock. If winning earns a payoff of $+1$ while losing earns a payoff of -1, RPS can be described by the following payoff matrix:

$$\begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array}\begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix} \end{array} \tag{1}$$

where the payoff in a given cell is that of the row strategy when playing against the column strategy.

Evolution is fundamentally a process of change over time, and so it is desirable to add a dynamic component to the payoff matrix. Strategies with above average payoff should increase in abundance while strategies with below average payoff should decrease. One popular approach, the "replicator equation," does so using differential equations [2,4]. The replicator equation describes deterministic evolutionary dynamics in a well-mixed, infinitely large population, and is defined as follows. Let A_{ij} be the payoff of strategy i playing against strategy j, and x_i be the fraction of players in the population using strategy i. Assuming there are N possible strategies, we have the constraint

$$\Sigma_{i=1}^{N} x_i = 1 \tag{2}$$

The "fitness" (or expected payoff) of an individual playing strategy i is given by

$$f_i = \Sigma_{j=1}^{N} A_{ij} x_j \tag{3}$$

and the replicator equation stipulates that

$$\dot{x}_i = x_i(f_i - \Phi) \tag{4}$$

where Φ is chosen as

$$\Phi = \Sigma_{j=1}^{N} x_j f_j \tag{5}$$

so that the constraint, Eq. (2), is satisfied.

In this paper, we use the replicator equation to study the RPS game because of RPS's wide range of applications in both biology and social science. The cyclical dynamics of RPS provide a natural model for many similarly cyclical natural phenomena. Two of the most widely cited examples involve mating patterns of the side-blotched lizard *Uta stansburiana* [5] and toxin and antidote production in mutant forms of the bacteria *Escherichia coli* [6]. In each case, there are three strategies, each of which out-competes another and is out-competed by the third. The result is cyclic dominance, with no strategy as the clear winner. This type of RPS dynamic has also been used to explain the "paradox of the plankton," in which ecosystems support a much greater degree of biodiversity than suggested by the number of ecological niches [7].

Outside of biology, RPS evolutionary dynamics have also been used to model a range of social interactions. Here, the dynamics describe a process of social learning via imitation rather than genetic evolution. Each person has a strategy, and people preferentially imitate the strategies of more successful others, ignoring the strategies chosen by the less successful. Thus natural selection operates on strategy abundances, producing a replicator dynamic identical to genetic evolution. One example of an RPS dynamic in human social interactions involves optional cooperative relationships [8–10]: selfish players invade a population of cooperative players, loners who abstain invade a population of selfish players, and cooperators invade a population of loners. Another example comes from opinion formation in political elections [11], where a set of candidates may each have arguments which expose weakness in another candidate, but are vulnerable to attacks from a third.

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In the present work, we add periodic variation in payoffs to the RPS game. These periodic effects could represent, for example, fitness changes caused by seasonal fluctuations in the weather, or earning changes caused by seasonal variation in consumer spending. This paper builds on previous work from our group [12,13]. In Ref. [12], we studied the linear stability of an RPS system with periodic coefficients. In Ref. [13], we extended the work by considering the nonlinear stability of a similar system. In the present work we consider the linear stability of a similar system which however contains a larger number of parameters. We find that by thus increasing the number of parameters, we are able to observe events which would be unlikely to occur in a system with few parameters. In particular we have discovered that the model supports the disappearance of tongues of resonant instability.

2 Model

We are interested in an extension of the RPS model where the payoffs vary periodically with time. More specifically we consider a RPS payoff matrix of the form:

$$\begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{P} & \text{S} \end{array} \\ \begin{pmatrix} 0 & -1+A_1\cos\omega t & 1+A_2\cos\omega t \\ 1+A_3\cos\omega t & 0 & -1+A_4\cos\omega t \\ -1+A_5\cos\omega t & 1+A_6\cos\omega t & 0 \end{pmatrix} \end{array} \tag{6}$$

Here, the strength and frequency of the variation can be manipulated by the ω and A_i parameters. This payoff matrix gives rise to three replicator equations given by Eq. (4) which can then be reduced to two equations on x_1 and x_2 by eliminating x_3 via the constraint, Eq. (2), $x_3 = 1 - x_1 - x_2$. The result is

$$\dot{x}_1 = x_1(1 - 2x_2 - x_1) + x_1 G_1 \cos\omega t \tag{7}$$

$$\dot{x}_2 = x_2(x_2 + 2x_1 - 1) + x_2 G_2 \cos\omega t \tag{8}$$

where

$$G_1 = A_2(1 - x_1 - x_2) + x_2[A_1 - x_1(A_1 + A_3)] + F \tag{9}$$

$$G_2 = A_4(1 - x_1 - x_2) + x_1[A_3 - x_2(A_1 + A_3)] + F \tag{10}$$

where

$$F = (x_1 + x_2 - 1)[x_1(A_2 + A_5) + x_2(A_4 + A_6)] \tag{11}$$

Setting $A_i = 0$, gives the original RPS model where the dynamics of the replicator equations (Eqs. (7) and (8)) can be described by the first integral

$$x_1 x_2(1 - x_1 - x_2) = \text{constant} \tag{12}$$

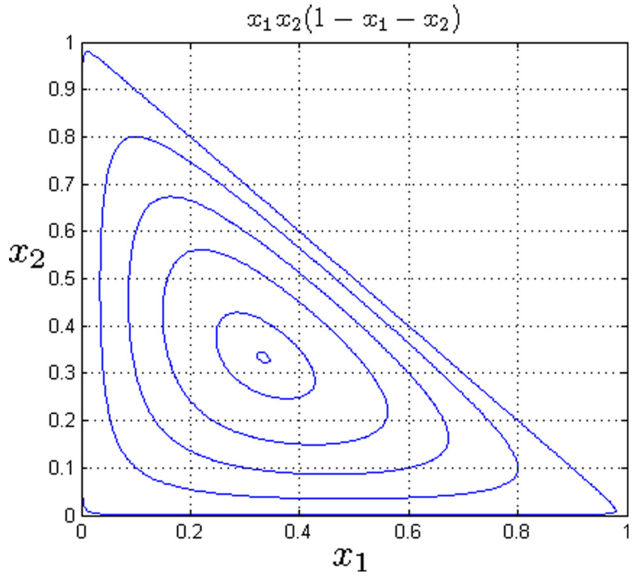

Fig. 1 Integral curves from Eq. (12). Each of these curves represents a motion which is periodic in time.

Equation (12) represents a family of curves, each of which corresponds to a motion which is periodic in time. In Fig. 1 we see the integral curves for various values of the constant in Eq. (12). We also find that for Eqs. (7) and (8), the points (1,0), (0,1) and (0,0) are equilibria and the lines $x_1 = 0$, $x_2 = 0$ and $x_1 + x_2 = 1$ are exact solutions. Note that $x_1 + x_2 = 1$ is equivalent to $x_3 = 0$ in view of Eq. (2). There is also an interior equilibrium point at $(1/3, 1/3)$.

The presence of the time-varying periodic terms $A_i \cos\omega t$ destroys the first integral, Eq. (12). In addition, for general values of the A_i these terms destroy the equilibrium at (1/3),(1/3). We wish to consider the case in which this equilibrium is preserved under the periodic forcing. From Eqs. (7) and (8), this will require that G_1 and G_2 vanish at $x_1 = x_2 = 1/3$. This turns out to require the following relationship between the A_i coefficients:

$$A_1 + A_2 = A_3 + A_4 = A_5 + A_6 \tag{13}$$

To satisfy Eq. (13), we set

$$A_3 = A_5 + A_6 - A_4 \tag{14}$$

$$A_1 = A_5 + A_6 - A_2 \tag{15}$$

This corresponds to the payoff matrix

$$\begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{P} & \text{S} \end{array} \\ \begin{pmatrix} 0 & -1+(A_5+A_6-A_2)\cos\omega t & 1+A_2\cos\omega t \\ 1+(A_5+A_6-A_4)\cos\omega t & 0 & -1+A_4\cos\omega t \\ -1+A_5\cos\omega t & 1+A_6\cos\omega t & 0 \end{pmatrix} \end{array} \tag{16}$$

and to the following governing differential equations:

$$\begin{aligned} \dot{x}_1 = x_1((A_6(x_2^2 - x_1x_2) + A_5(x_2(1 - x_1) + x_1^2 - x_1) \\ + A_4(x_2^2 + x_2(2x_1 - 1)) + A_2(x_2(2x_1 - 2) \\ + x_1^2 - 2x_1 + 1))\cos\omega t - 2x_2 - x_1 + 1) \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{x}_2 = x_2((A_6(x_2^2 - (x_1 + 1)x_2 + x_1) + A_5(x_1^2 - x_1x_2) \\ + A_4(x_2^2 + (2x_1 - 2)x_2 + 1 - 2x_1) \\ + A_2(2x_1x_2 + x_1^2 - x_1))\cos\omega t + x_2 + 2x_1 - 1) \end{aligned} \tag{18}$$

In our previous work [12,13], we showed that for the case of $A_1 = -A_2 = A$, $A_3 = A_4 = A_5 = A_6 = 0$ the interior equilibrium point $(1/3, 1/3)$ changed stability for resonant values of the parameters ω and A. Using perturbation theory, we were able to detect tongues of instability in the parameter space as well as describe the nonlinear behavior in the different regions of the tongues.

In this work we seek to investigate the existence of such tongues for the more general case, Eqs. (14) and (15), in which $A_1 = A_5 + A_6 - A_2$ and $A_3 = A_5 + A_6 - A_4$.

3 Subharmonic Resonance

We begin by investigating the linear stability of the interior equilibrium point. First we move the interior equilibrium point to the origin for convenience.

$$x_1 = x + \frac{1}{3}, \quad x_2 = y + \frac{1}{3} \tag{19}$$

Then substitute Eqs. (19) into Eqs. (17) and (18)

$$\begin{aligned}\dot{x} = \frac{1}{9}[&\cos\omega t\{((9x+3)y^2 + (1-9x^2)y - 3x^2 - x)A_6 \\ &+ ((-9x^2+3x+2)y + 9x^3 - 3x^2 - 2x)A_5 \\ &+ ((9x+3)y^2 + (18x^2+9x+1)y + 6x^2 + 2x)A_4 \\ &+ ((18x^2-6x-4)y + 9x^3 - 3x^2 - 2x)A_2\} \\ &+ y(-18x-6) - 9x^2 - 3x]\end{aligned} \tag{20}$$

$$\begin{aligned}\dot{y} = \frac{1}{9}[&\cos\omega t\{(9y^3 + (-9x-3)y^2 + (3x-2y) + 2x)A_6 \\ &+ ((-9x-3)y^2 + (9x^2-)y + 3x^2 + x)A_5 \\ &+ (9y^3 + (18x-3)y^2 + (-6x-2y) - 4x)A_4 \\ &+ ((18x+6)y^2 + (9x^2+9x+2)y + 3x^2 + x)A_2\} \\ &+ 9y^2 + y(18x+3) + 6x]\end{aligned} \tag{21}$$

For a linear stability analysis, we linearize Eqs. (20) and (21)

$$\dot{x} = \frac{((y-x)A_6 + (2y-2x)A_5 + (y+2x)A_4 + (-4y-2x)A_2x)\cos\omega t - 6y - 3x}{9} \tag{22}$$

$$\dot{y} = \frac{((2x-2y)A_6 + (x-y)A_5 + (-2y-4x)A_4 + (2y+x)A_2)\cos\omega t + 3y + 6x}{9} \tag{23}$$

Now, we transform this system of first-order ODEs into a second-order ODE for convenience in eliminating secular terms in the upcoming perturbation method. We find

$$f_1\ddot{x} + f_2\dot{x} + f_3 x = 0 \tag{24}$$

where

$$f_1 = -9((A_6 + 2A_5 + A_4 - 4A_2)\cos\omega t - 6) \tag{25}$$

$$\begin{aligned}f_2 =& 18(A_6 + A_5)\cos\omega t - 9\omega(A_6 + 2A_5 + A_4 - 4A_2)\sin\omega t \\ &- 3(A_6 + A_5)(A_6 + 2A_5 + A_4 - 4A_2)\cos^2\omega t\end{aligned} \tag{26}$$

$$\begin{aligned}f_3 =& 18 + 3(A_6 - 4A_5 - 5A_4 + 8A_2)\cos\omega t \\ &- 9\omega(A_6 + 2A_5 - A_4)\sin\omega t + (-A_6^2 + (-A_5 + A_4 + 8A_2)A_6 \\ &+ 2A_5^2 + (11A_4 - 8A_2)A_5 + 2A_4^2 - 16A_2A_4 + 8A_2^2)\cos^2\omega t \\ &- (A_6 + 2A_5 + A_4 - 4A_2)(A_2A_6 + A_4A_5 - A_2A_4)\cos^3\omega t\end{aligned} \tag{27}$$

We may now use a perturbation method to determine the stability of the interior equilibrium which has now been moved to the origin, under the assumption of small forcing amplitudes. To use the perturbation method we make a change of variables $\tau = \omega t$ and denote $'$ as a derivative with respect to τ. We also write $A_i \to \epsilon A_i$. This gives, neglecting terms of $O(\epsilon^2)$

$$g_1 x'' + g_2 x' + g_3 x = O(\epsilon^2) \tag{28}$$

where

$$g_1 = 54\omega^2 - 9\omega^2\epsilon(A_6 + 2A_5 + A_4 - 4A_2)\cos\tau \tag{29}$$

$$g_2 = 18\omega\epsilon(A_6 + A_5)\cos\tau - 9\omega^2\epsilon(A_6 + 2A_5 + A_4 - 4A_2)\sin\tau \tag{30}$$

$$\begin{aligned}g_3 =& 18 + 3\epsilon(A_6 - 4A_5 - 5A_4 + 8A_2)\cos\tau \\ &- 9\epsilon\omega(A_6 + 2A_5 - A_4)\sin\tau\end{aligned} \tag{31}$$

To begin with, we determine the resonant value of ω at $O(\epsilon)$ by setting

$$x = x_0 + \epsilon x_1 + O(e^2) \tag{32}$$

Substituting Eq. (32) into Eq. (28) and collecting terms gives

$$x_0'' + \frac{x_0}{3\omega^2} = 0 \tag{33}$$

$$x_1'' + \frac{x_1}{3\omega^2} = H_1 x_0'' + H_2 x_0' + H_3 x_0 \tag{34}$$

where

$$H_1 = \frac{(A_6 + 2A_5 + A_4 - 4A_2)\cos\tau}{6} \tag{35}$$

$$H_2 = \frac{\omega(A_6 + 2A_5 + A_4 - 4A_2)\sin\tau + (-2A_6 - 2A_5)\cos\tau}{6\omega} \tag{36}$$

$$H_3 = \frac{\omega(3A_6 + 6A_5 - 3A_4)\sin\tau + (-A_6 + 4A_5 + 5A_4 - 8A_2)\cos\tau}{18\omega^2} \tag{37}$$

From Eq. (33), we see that x_0 will have a solution with frequency $1/\sqrt{3}\omega$ whereupon the right hand side of Eq. (34) will have terms with frequencies

$$1 \pm \frac{1}{\sqrt{3}\omega} \tag{38}$$

Resonant values of ω will correspond to forcing frequencies Eq. (38) which are equal to natural frequencies of the homogeneous x_1 equation, i.e., to $1/\sqrt{3}\omega$. This gives that

$$\omega = \frac{2}{\sqrt{3}} \quad \text{(resonance)} \tag{39}$$

This value of ω corresponds to the largest resonance tongue. There are an infinitude of smaller tongues which would emerge from the perturbation method if we were to continue it to $O(\epsilon^2)$ and higher. These have been shown [12] to be of the form $\omega_0 = 2/(n\sqrt{3})$ for $n = 2, 3, \ldots$ but will not concern us in this paper.

In order to investigate the nature of the dynamical behavior in the neighborhood of the resonance, Eq. (39), we define two time scales ξ and η

$$\xi = \tau, \quad \eta = \epsilon\tau \tag{40}$$

and we consider x to be a function of ξ and η, whereupon the chain rule gives

$$x' = x_\xi + \epsilon x_\eta \tag{41}$$

$$x'' = x_{\xi\xi} + 2\epsilon x_{\xi\eta} + \epsilon^2 x_{\eta\eta} \tag{42}$$

We detune ω off of the resonance Eq. (39)

$$\omega = \frac{2}{\sqrt{3}} + k_1\epsilon + \cdots \tag{43}$$

and expand $x = x_0 + \epsilon x_1 + \cdots$. Substituting Eqs. (41) and (42) and these expansions into Eq. (28) and collecting terms, we obtain

$$x_{0\xi\xi} + \frac{1}{4}x_0 = 0 \tag{44}$$

$$x_{1\xi\xi} + \frac{1}{4}x_1 = -2x_{0\xi\eta} + h_1 x_{0\xi\xi} + h_2 x_{0\xi} + h_3 x_0 + \frac{\sqrt{3}}{4}k_1 x_0 \tag{45}$$

where the functions h_i in Eq. (45) are the same as the functions H_i in Eq. (28) with τ replaced by ξ and ω replaced by $2/\sqrt{3}$.

We take the solution of Eq. (44) in the form

$$x_0 = a(\eta)\cos\frac{\xi}{2} + b(\eta)\sin\frac{\xi}{2} \tag{46}$$

We substitute the expression for x_0 Eq. (46) into the x_1 Eq. (45), and remove secular terms, giving the slow flow

$$\frac{\partial a}{\partial \eta} = a\left(\frac{A_4 - A_5}{8\sqrt{3}}\right) + b\left(-\frac{\sqrt{3}}{4}k_1 + \frac{A_2 - A_6}{12} + \frac{A_4 - A_5}{24}\right) \tag{47}$$

$$\frac{\partial b}{\partial \eta} = -b\left(\frac{A_4 - A_5}{8\sqrt{3}}\right) + a\left(\frac{\sqrt{3}}{4}k_1 + \frac{A_2 - A_6}{12} + \frac{A_4 - A_5}{24}\right) \tag{48}$$

Equations (47) and (48) are a constant coefficient linear system with the following eigenvalues:

$$\pm\frac{1}{12}\sqrt{-27k_1^2 + (A_2 - A_6)^2 + (A_4 - A_5)^2 + (A_2 - A_6)(A_4 - A_5)} \tag{49}$$

For given parameters A_2, A_4, A_5, A_6, the equilibrium point $a = b = 0$ will be either unstable (exponential growth) or stable (quasi-periodic motion) depending respectively on whether the eigenvalues, Eq. (49), are real or imaginary. The transition between stable and unstable will correspond to zero eigenvalues, given by the condition

$$27k_1^2 = (A_2 - A_6)^2 + (A_4 - A_5)^2 + (A_2 - A_6)(A_4 - A_5) \tag{50}$$

Equation (50) will yield two values of k_1, let's call them $k_1 = \pm Q$, which from Eq. (43) plot as two straight lines in the $\omega - \epsilon$ plane, representing the boundaries of the 2:1 subharmonic resonance tongue, see Fig. 2. Inside this tongue the equilibrium is unstable due to parametric resonance

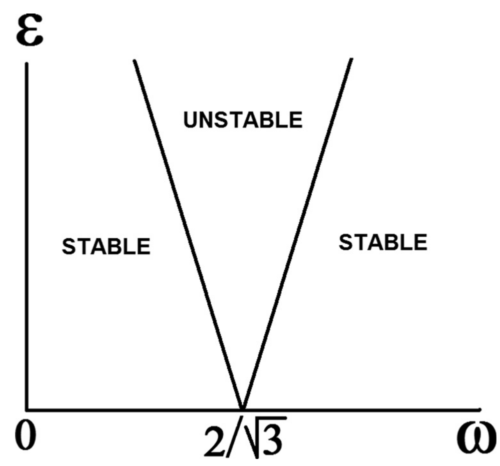

Fig. 2 2:1 subharmonic resonance tongue, Eq. (51). The RPS equilibrium point at $x_1 = x_2 = 1/3$ is linearly unstable for parameters inside the tongue. The presence of nonlinearities detunes the resonance and prevents unbounded motions which are predicted by the linear stability analysis.

$$\omega = \frac{2}{\sqrt{3}} \pm Q\epsilon, \quad Q = \frac{\sqrt{(A_2 - A_6)^2 + (A_4 - A_5)^2 + (A_2 - A_6)(A_4 - A_5)}}{\sqrt{27}} \tag{51}$$

4 Disappearing Tongue

In the special case that $A_2 = A_6$ and $A_4 = A_5$, we see from Eq. (51) that $Q = 0$ and the tongue has closed up, at least to $O(\epsilon)$. For these parameter values we have from Eqs. (14) and (15)

$$A_1 = A_4 = A_5 \equiv \alpha\epsilon, \quad A_2 = A_3 = A_6 \equiv \beta\epsilon \tag{52}$$

so that the payoff matrix, Eq. (6), becomes

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 + \alpha\epsilon\cos\omega t & 1 + \beta\epsilon\cos\omega t \\ 1 + \beta\epsilon\cos\omega t & 0 & -1 + \alpha\epsilon\cos\omega t \\ -1 + \alpha\epsilon\cos\omega t & 1 + \beta\epsilon\cos\omega t & 0 \end{pmatrix} \end{array} \tag{53}$$

where $\omega = 2/\sqrt{3}$, and the linearized differential Eqs. (22) and (23) become

$$\dot{x} = \frac{(y\alpha\epsilon - (x + y)\beta\epsilon)\cos\omega t - 2y - x}{3} \tag{54}$$

$$\dot{y} = \frac{(-(x + y)\alpha\epsilon + x\beta\epsilon)\cos\omega t + y + 2x}{3} \tag{55}$$

From Floquet theory [14,15] we know that on the transition curves which define the two sides of the tongue, i.e., which separate regions of stability from regions of instability, there exists a periodic solution having frequency $\omega/2$ (a "subharmonic"). To prove that the tongue has truly disappeared (rather than approximately so as in perturbation theory), we must show that there COEXISTS two linearly independent solutions having frequency $\omega/2$. To make this easier to consider, define a new subharmonic time scale $T = (\omega/2)t = t/\sqrt{3}$. Then Eqs. (54) and (55) become

$$\frac{1}{\sqrt{3}}\frac{dx}{dT} = \frac{(y\alpha\epsilon - (x + y)\beta\epsilon)\cos 2T - 2y - x}{3} \tag{56}$$

$$\frac{1}{\sqrt{3}}\frac{dy}{dT} = \frac{(-(x+y)\alpha\epsilon + x\beta\epsilon)\cos 2T + y + 2x}{3} \tag{57}$$

Here, we must show that there exists two linearly independent solutions with frequency 1 in time variable T. For example, when $\alpha = \beta = 0$, there are two linearly independent solutions with frequency 1

$$x = \sqrt{3}\cos T - \sin T, \quad y = 2\sin T \tag{58}$$

and

$$x = -2\sin T, \quad y = \sqrt{3}\cos T + \sin T \tag{59}$$

That is, we are forcing the system at twice its natural frequency. The idea here is that there normally exists a solution of frequency 1 on each transition curve. In order to show that there is no tongue, we have to show that the two transition curves are coincident. In fact we claim that the two transition curves correspond to $k_1 = Q = 0$, that is, to a single vertical line in the $\omega - \epsilon$ plane, going through the point $\omega = 2/\sqrt{3}, \epsilon = 0$. Equations (56) and (57) correspond to such a vertical line, and so we want to show that there are two linearly independent solutions to these equations.

Numerical simulations of Eqs. (56) and (57) have shown that this result is valid to all orders of ϵ, i.e., Eqs. (56) and (57) exhibit a frequency 1 solution for all nontrivial initial conditions, regardless of the values of α, β or ϵ. That is, the tongue really does close up and the instability disappears. Moreover, numerical evidence shows that all the other tongues in the $\epsilon - \omega$ plane (which emanate from points on the ω-axis at $\omega = 2/(n\sqrt{3})$, see Ref. [12]) also close up and disappear.

We supplement these numerical results with the following:

Theorem. *All nontrivial solutions to Eqs. (56) and (57) are periodic with frequency 1.*

Proof: We assume a solution to Eqs. (54) and (55) in the form ("variation of parameters")

$$x = u(\sqrt{3}\cos T - \sin T) + v(-2\sin T) \tag{60}$$

$$y = u(2\sin T) + v(\sqrt{3}\cos T + \sin T) \tag{61}$$

where u and v are functions of T to be found. Note that $(u = 1, v = 0)$ gives Eq. (58), while $(u = 0, v = 1)$ gives Eq. (59). Substituting Eqs. (60) and (61) into Eqs. (56) and (57) gives the following Eqs. on u and v:

$$\sqrt{3}\frac{du}{dT} = \epsilon\cos 2T(-\beta u + (\alpha - \beta)v) \tag{62}$$

$$\sqrt{3}\frac{dv}{dT} = \epsilon\cos 2T((\beta - \alpha)u - \alpha v) \tag{63}$$

Next we define new time variable

$$dz = \frac{\epsilon\cos 2T}{\sqrt{3}}dT \Rightarrow z = \frac{\epsilon\sin 2T}{2\sqrt{3}} \tag{64}$$

which gives the following constant coefficient linear system on u, v:

$$\frac{d}{dz}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\beta & \alpha - \beta \\ \beta - \alpha & -\alpha \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \tag{65}$$

The matrix in Eq. (65) has eigenvalues

$$\lambda = -\left(\frac{\alpha+\beta}{2}\right) \pm i\frac{\sqrt{3}}{2}(\beta - \alpha) \tag{66}$$

Thus, the general solution to Eq. (65) involves a linear combination of terms of the form

$$\exp\left(\frac{\alpha+\beta}{2}z\right)\begin{bmatrix} \sin \\ \cos \end{bmatrix}\frac{\sqrt{3}}{2}(\beta - \alpha)z \tag{67}$$

Therefore, since z is a π-periodic function of T, we see that u and v also have period π in time T. Then from Eqs. (60) and (61), it follows that x and y have period 2π in T, since the product of a π-periodic function and a 2π-periodic function has period 2π. Q.E.D.

This phenomenon has been observed in various other parametric excitation problems and has been referred to as "coexistence" [15–17].

5 Conclusions

From a dynamical systems point of view, we may summarize our findings as follows: The original RPS system, with payoff matrix, Eq. (1), and no forcing, exhibits an equilibrium at (1/3,1/3) which is stable (Fig. 1). With the addition of forcing, there will generally be a 2:1 subharmonic resonance region in parameter space in which the equilibrium becomes unstable (Fig. 2). In the present work, we have shown that this tongue may be absent or very small if the forcing parameters are chosen appropriately.

In the case that the equilibrium is linearly unstable, the presence of nonlinearities detunes the resonance (because the frequency of the motion changes as the amplitude increases) and prevents the unbounded motions which are predicted by the linear stability analysis. The resulting unstable motion is either quasi-periodic or chaotic [13].

From a biological and social point of view, the presence of periodic forcing in RPS can lead to quasi-periodic or chaotic oscillations, such as those observed in the range of biological and social applications described above in the introduction: seemingly stochastic fluctuations in strategy abundances need not necessarily arise from a stochastic process, as we have shown in earlier work [12,13]. The findings of the current paper have further implications. Depending on the choice of forcing parameters, it is possible to reduce or even eliminate quasi-periodic motion. Thus, if one was designing an organization, community or political system where stability was desired, this effect could be achieved by properly tuning the degree of periodic forcing. A similar logic applies to biological systems. If the forcing coefficients were themselves subject to natural selection, evolution might favor coefficients that eliminate the tongue and result in stable population abundances.

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