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A SIMPLIFIED MODEL OF COUPLED RELAXATION OSCILLATORS*

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Abstract—In this paper we analyze a simplified model of a pair of coupled relaxation oscillators. The simplified model employs a two-dimensional phase space with jump conditions to approximate the behavior in the full four-dimensional phase space. The stability of the phase-locked modes of oscillation for the simplified system is obtained using phase plane techniques and compared to that previously obtained for the full system using asymptotic and numerical methods. It is shown that these results differ qualitatively over a region of the parameter space, and Lyapunov's direct stability method is used to explain this shortcoming of the model.

INTRODUCTION

Recent efforts to understand the dynamics of systems of coupled relaxation oscillators have led to some results for two oscillators with special kinds of coupling [1, 2]; but also to the realization that the problem is difficult, generalizations of special results may not be available, and there is little hope for solving larger systems. Even numerical investigations are difficult and expensive since very small time steps must be used during the rapid jumps that occur. While this last problem may be overcome through the use of numerical methods based on exponential interpolants [3], it still seems that a simpler model of a relaxation oscillator could be of great help in developing a feel for the behavior of coupled relaxation oscillators. Here we propose such a model and analyze some of its capabilities and limitations. We will see that the simplified model works well in much of the parameter space, but surprisingly fails to predict the proper behavior in certain regions. Application of direct stability methods will help us to understand the reason for this failure.

DERIVATION OF THE SIMPLIFIED OSCILLATOR

The type of oscillator which we wish to simplify is typified by the van der Pol (vdp) relaxation oscillator [4]

$$\varepsilon x'' + \varphi_{\mathrm{vdp}}(x)x' + x = 0, \qquad \varphi_{\mathrm{vdp}}(x) = x^2 - 1. \tag{1}$$

An equivalent system of first order equations in Lienard variables can be obtained as follows. Introduce a new variable u such that $u' = \varepsilon x'' + \varphi(x)x'$. Then equation (1) immediately becomes

$$u' = -x \tag{2a}$$

while integrating the expression defining u and solving for x' gives

$$x' = (u - \Phi_{\mathrm{vdp}}(x))/\varepsilon, \qquad \text{where } \Phi_{\mathrm{vdp}}(x) = -x + x^3/3 = \int \varphi_{\mathrm{vdp}}(x)\,dx. \tag{2b}$$

The flow in the x, u phase space is shown in Fig. 1a. Note the presence of the stable limit

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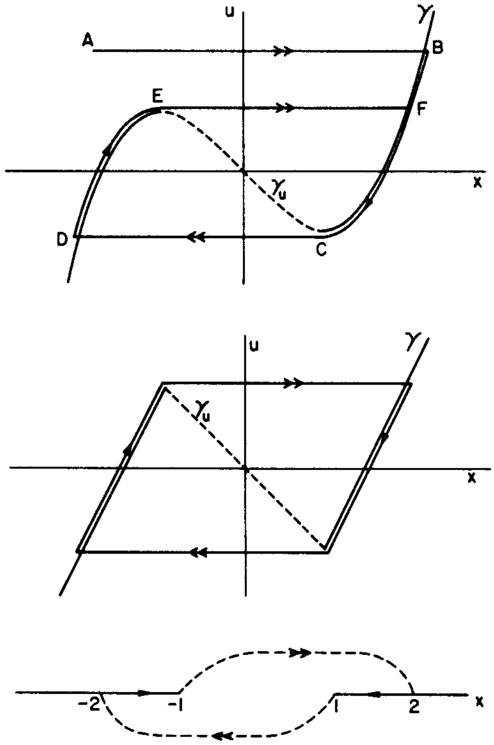

Fig. 1. (a) Phase portrait of the van der Pol relaxation oscillator using Liénard variables. (b) Phase portrait of the piecewise linear relaxation oscillator. (c) Flow and jumps in the reduced phase space for the simplified relaxation oscillator. Single arrows indicate slow segments of the limit cycle while double arrows indicate rapid jumps.

cycle (EFDC) consisting of "slow" segments close to the curve $\gamma_{vdp} = \{(x, u) \mid u = \Phi_{vdp}(x)\}$ and "fast" horizontal jumps. The first simplification, introduced by Levinson [5] (and later used by Levi [6] and Belair and Holmes [1]), is to employ a piecewise linear (pwl) version of γ as shown in Fig. 1b. Thus let

$$\Phi_{pwl}(x) = \begin{cases} 2x + 3, & x \leqslant -1 \\ -x, & |x| < 1 \\ 2x - 3, & x \geqslant 1. \end{cases} \tag{3}$$

(We choose the slope $\varphi_{pwl}(x) = 2$ for $|x| > 1$ so that the extreme values of x attained on the resulting limit cycle agree with those for the original van de Pol equation; i.e. the oscillator decays to ± 1 then jumps to ∓ 2.)

Next we note that in the limit as $\varepsilon \to 0$ solutions are immediately attracted to γ_{pwl} and the jumps become instantaneous. The flow is constrained to the stable branches of γ_{pwl} connected by instantaneous horizontal jumps. Projecting this one-dimensional flow onto the x-axis, the reduced phase becomes the x-axis with the segment $|x| < 1$ removed. The flow on the reduced phase space (where $u = 2x \pm 3$) is governed by

$$u' = 2x' = -x \quad \Rightarrow \quad x' = -x/2. \tag{4}$$

When a trajectory reaches $x = \pm 1$, it instantaneously jumps to $x = \mp 2$, and we have obtained a discontinuous approximation to the limit cycle of the oscillator [7]. This system consisting of a linear flow on $\mathbf{R} \backslash (-1, 1)$ [i.e. the complement of $(-1, 1)$ in $\mathbf{R}$] with jumps (see Fig. 1c) is the simplified relaxation oscillator. The simplified model gives a good qualitative description of the slow decay and rapid jumps characterizing the free response of the relaxation oscillator as shown in Fig. 2. (Note that the time axes in Fig. 2 are drawn to different scales since the period of the simplified oscillator, $T_{\text{simp}} = 2\ln 2 \approx 1.386$, does not agree quantitatively with the period, $T_{\text{vdp}} = 2 - 2\ln 2 + O(\varepsilon^{2/3}) \approx 1.614$, of the van der Pol relaxation oscillator.) In another paper we will show that the simplified model also gives a good description of the periodic response of a harmonically forced relaxation oscillator. Here we focus on the dynamics of coupled oscillators, concentrating on the stability of phase locked modes.

THE REDUCED SYSTEM: A PAIR OF SIMPLIFIED OSCILLATORS

Belair and Holmes [1] used a version of the simplified relaxation oscillator to obtain the stability of phase-locked modes for a pair of relaxation oscillators with linear diffusive velocity coupling:

$$\begin{aligned} x'' + \varphi_{\text{pwl}}(x)x' + x &= \beta(y' - x') \\ y'' + \varphi_{\text{pwl}}(y)y' + \omega y &= \beta(x' - y'), \qquad \beta \geqslant 0. \end{aligned} \tag{5}$$

They showed that a reduced model of the system (consisting of two simplified oscillators) is sufficient to describe the stability of the phase-locked modes for the full piecewise linear system (which is qualitatively equivalent to the full system of two van der Pol oscillators).

Can a reduced model of the relaxation oscillator also give an accurate qualitative picture of the behavior of a pair of van der Pol relaxation oscillators with similar coupling in the displacements? Let us answer this question by analyzing the reduced system and comparing the results with those already published [2] for the full system

$$x'' + \varphi_{\text{vdp}}(x)x' + x = \alpha(y - x) \tag{6a}$$

$$y'' + \varphi_{\text{vdp}}(y)y' + y = \alpha(x - y). \tag{6b}$$

The most significant aspect of the simplified relaxation oscillator for this analysis is that its phase space is one-dimensional. This allows for the application of phase plane techniques to a reduced system of two coupled simplified relaxation oscillators. The equations of motion governing such a system are derived by the same procedure that led to (4) above. We replace the first two terms in (6a) and (6b) by u' and v' respectively, and restrict the

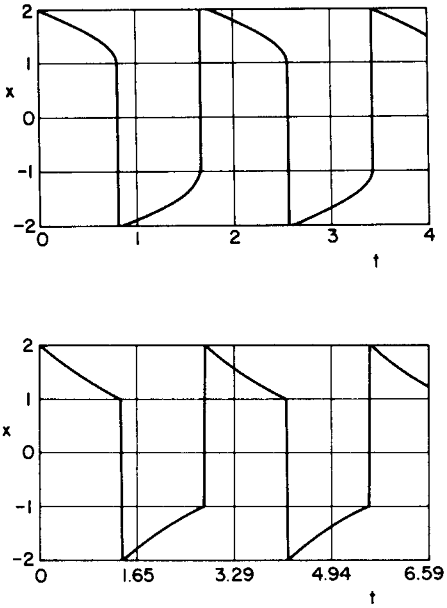

Fig. 2. (a) Limit cycle of equation (1) with $\varepsilon = 0.0001$. (b) Behavior of the simplified oscillator.

flow to the slow manifold given by $u = \Phi_{pwl}(x)$ and $v = \Phi_{pwl}(y)$ (so that $u' = 2x'$ and $v' = 2y'$) to obtain

$$\begin{aligned} x' &= -[x + \alpha(y - x)]/2 \\ y' &= -[y + \alpha(x - y)]/2 \end{aligned} \tag{7}$$

where we have assumed diffusive coupling in the displacements as above. The phase space for the reduced system, given by $\{\mathbf{R}\backslash(-1,1) \times \mathbf{R}\backslash(-1,1)\}$, is shown in Fig. 3, and each variable is subject to the jump condition described earlier.

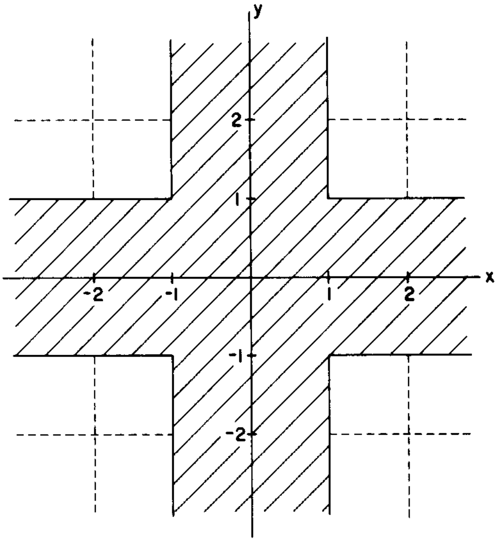

Fig. 3. Phase space for the reduced system showing jump lines (solid), landing lines (broken), and the excluded region (shaded).

To determine the dynamics of the system, we first examine the "background flow" (i.e. the flow in the full x, y plane ignoring the jumps), then introduce the jump conditions to determine the possible steady-state behaviors and their stability. The background flow is a simple linear flow with a single fixed point at the origin. The associated eigenvalues and eigenvectors are

$$\lambda_1 = \frac{-1}{2}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad \lambda_2 = \frac{-1-2\alpha}{2}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{8}$$

and the flow is shown in Fig. 4 for different values of the coupling parameter α. Now we can begin to consider how the jumps determine the eventual behavior of the system. In Fig. 4a, the two bold segments (E, E′, F, F′, E) on the line $y = x$ correspond to the in-phase mode where the two oscillators jump and decay in unison with identical time histories, while the bold segments (a, a′, b, b′, a) on the line $y = -x$ correspond to the out-of-phase mode where the oscillators jump and decay in unison but with opposite sign resulting in a 180° phase difference. Note that for $\alpha < -1/2$ (Fig. 4c), $\lambda_2 > 0$ and the equilibrium is a saddle point. Trajectories originating on the line $y = -x$ go off to ∞, never encountering a jump line, and the out-of-phase mode does not exist. In fact almost all trajectories are attracted to this unstable direction (for example trajectory A, B, B′, C in Fig. 4c), so the in-phase mode (E, E′, F, F′, E) exists but is unstable. Similarly, by following the appropriate trajectories and jumps (such as those indicated in Figs 4a and b), the stability results shown in Fig. 5 are obtained.

The stability results for the full system of two coupled van der Pol relaxation oscillators, obtained by a perturbation method and verified by numerical integration [2], are given in Fig. 6. Note that they are qualitatively different from those for the reduced system. Therefore, the reduced system does not give an accurate description of the dynamics of

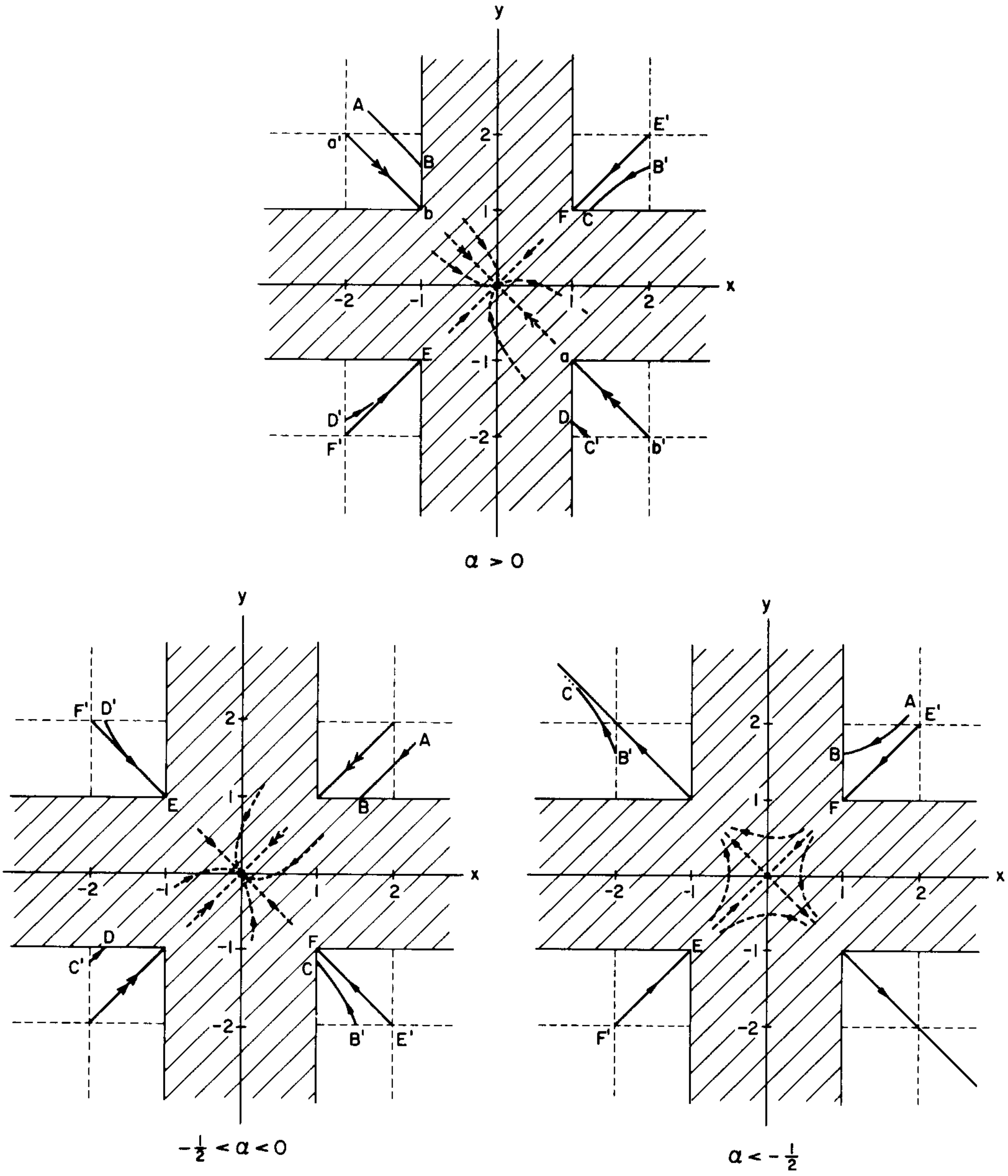

Fig. 4. Linear flow in the x, y plane (with jump/landing lines superposed) for (a) $\alpha > 0$, (b) $-1/2 < \alpha < 0$, (c) $\alpha < -1/2$. Trajectories follow alphabetical order with primes indicating position following a jump.

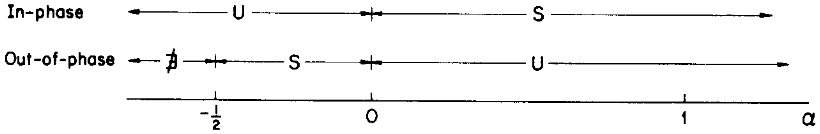

Fig. 5. Stability of the in-phase mode for the reduced system of two simplified oscillators.

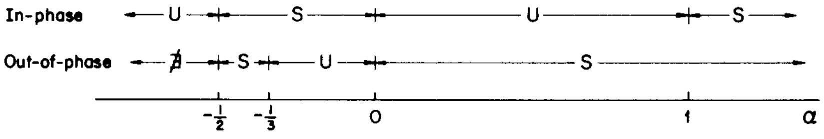

Fig. 6. Stability of the in-phase mode for the full system of two van der Pol relaxation oscillators.

the full system in this case. From a modeler's point of view, it is important to understand why the reduced system fails to give the "correct" stability results. We can shed some light on this question by considering the results of Lyapunov's direct stability method.

IMPLICATIONS OF DIRECT STABILITY METHODS

The stability of the in-phase mode $x(t;\varepsilon) \equiv y(t;\varepsilon) \equiv \xi(t;\varepsilon)$ [where $\xi(t;\varepsilon)$ represents the limit cycle] has been shown in [2] to be governed by the stability of the trivial solution of the variational equation

$$\varepsilon\eta'' + \varphi(\xi)\eta' + (1 + 2\alpha)\eta = 0. \tag{9}$$

See the Appendix for a derivation of this result.

To determine the stability of $\eta \equiv 0$, let us employ Lyapunov's direct method (LaSalle and Lefshetz [8]) and seek a function $F(\eta, \eta')$ which satisfies the following properties in some neighborhood of the origin:

(1) $F(\eta, \eta')$ is real and has continuous first partial derivatives.
(2) $F(0, 0) = 0$.
(3) F is positive definite, i.e. $F(\eta, \eta') > 0$ for all $(\eta, \eta') \neq (0, 0)$.
(4) F' is negative semidefinite, i.e. $F' \leqslant 0$ for all (η, η').

If such a Lyapunov function exists, then the origin of the phase plane of the variational equation and, consequently, the in-phase mode are stable. Consider the following candidate Lyapunov function:

$$F(\eta, \eta') = [\varepsilon\eta'^2 + (1 + 2\alpha)\eta^2]/2. \tag{10}$$

Note that F satisfies properties 1–3 (for $\alpha > -1/2$) and its time derivative is given by

$$F'(\eta, \eta') = \eta'[\varepsilon\eta'' + (1 + 2\alpha)\eta] = -\varphi(\xi)\eta'^2. \tag{11}$$

Therefore, the in-phase mode is stable if $\varphi(\xi)$ is never negative. For both the van der Pol relaxation oscillator and the piecewise linear relaxation oscillator, $\varphi(\xi)$ is negative if and only if $|\xi| < 1$, and this occurs only during the portion of the limit cycle where the rapid jump occurs. Thus if we apply the discontinuous approximation of the simplified oscillator (Fig. 2), the jump becomes instantaneous and the region for which $|\xi| < 1$ becomes excluded from consideration in (11). This gives the misleading result that the in-phase mode is stable for all $\alpha > -1/2$ and, like our foregoing analysis of the simplified model, it entirely misses the region of instability between $\alpha = 0$ and $\alpha = 1$ which has been found by asymptotic and numerical methods (cf. Fig. 6).

What went wrong? By invoking the simplified oscillator's jump, we ignored a brief time interval in which $\varphi(\xi)$ is positive. It is incorrect to draw the stability conclusion based on the non-positiveness of $F(\eta, \eta')$ in (11), since the brief interval during which $\varphi(\xi)$ is positive is accompanied by large velocities η' (see [2]) giving an appreciable change in F during the jump ($\Delta F = \int \varphi(\xi)\eta'^2 \, dt$).

CONCLUSION

While a simplified model of relaxation oscillators involving a discontinuous version of the limit cycle can be useful and effective in studying some problems [1], we have presented an example where it produces misleading stability predictions in a portion of the parameter space. This discrepancy occurs because details of the rapid jumps are lost when employing a discontinuous approximation of the limit cycle. By using Lyapunov's direct method, we have shown that these jumps can play a crucial role in determining the stability of the in-phase mode for a pair of coupled relaxation oscillators. This result reinforces our earlier finding [2] that the shape of the limit cycle has a significant effect on the stability of phase-locked modes in systems of coupled self-sustaining oscillators.

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APPENDIX: DERIVATION OF THE VARIATIONAL EQUATION (9)

To determine stability, we examine the time evolution of small variations from the in-phase mode $x(t;\varepsilon) \equiv y(t;\varepsilon) \equiv \xi(t;\varepsilon)$. Thus let

$$x = \xi + \delta x, \qquad y = \xi + \delta y. \tag{A1}$$

Substituting into equation (6) and retaining linear terms yields

$$\varepsilon(\delta x)'' + \varphi(\xi)(\delta x)' + (\varphi_\xi \xi' + 1)\delta x = \alpha(\delta y - \delta x) \tag{A2}$$

$$\varepsilon(\delta y)'' + \varphi(\xi)(\delta y)' + (\varphi_\xi \xi' + 1)\delta y = \alpha(\delta x - \delta y) \tag{A3}$$

where $\varphi_\xi = \mathrm{d}\varphi/\mathrm{d}\xi$. To uncouple and simplify these equations, we change variables according to

$$\theta' = \delta x + \delta y, \qquad \eta' = \delta x - \delta y. \tag{A4}$$

Substituting into (A3) and (A4) gives

$$\varepsilon\theta''' - \varphi(\xi)\theta'' + (1 + \varphi_\xi \xi')\theta' = 0 \tag{A5}$$

$$\varepsilon\eta''' - \varphi(\xi)\eta'' + (1 + 2\alpha + \varphi_\xi \xi')\eta' = 0 \tag{A6}$$

which can be integrated to yield

$$\varepsilon\theta'' - \varphi(\xi)\theta' + \theta = k_1 \tag{A7}$$

$$\varepsilon\eta'' - \varphi(\xi)\eta' + (1 + 2\alpha)\eta = k_2 \tag{A8}$$

where k_1 and k_2 are constants of integration. Note that the particular solutions of (A7) and (A8) are constants ($\theta_p = k_1$, $\eta_p = k_2/\{1 + 2\alpha\}$) which make no contribution to the variations $\delta x = (\theta' + \eta')/2$ and $\delta y = (\theta' - \eta')/2$. Therefore, we can consider the homogeneous variational equations

$$\varepsilon\theta'' - \varphi(\xi)\theta' + \theta = 0 \tag{A7}$$

$$\varepsilon\eta'' - \varphi(\xi)\eta' + (1 + 2\alpha)\eta = 0 \tag{A8}$$

where the right-hand sides have been set to zero. As pointed out in [2], (A7) is associated with the variational equation governing a single (uncoupled) stable limit cycle oscillator and cannot lead to instability. Therefore the stability of the in-phase mode is determined by the stability of (A8).