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# Coexistence phenomenon in autoparametric excitation of two degree of freedom systems

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**Abstract**

Coexistence phenomenon refers to the absence of expected tongues of instability in parametrically excited systems. In this paper we obtain sufficient conditions for coexistence to occur in the generalized Ince equation

$$(1 + a_1 \cos t + a_2 \cos 2t)\ddot{v} + (b_1 \sin t + b_2 \sin 2t)\dot{v} + (\delta + c_1 \cos t + c_2 \cos 2t)v = 0.$$

The results are applied to the stability of motion of a non-linear normal mode, the *x*-mode, in a class of conservative two degree of freedom systems.
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*Keywords:* Parametric excitation; Coexistence; Resonance; Stability

## 1. Introduction

### 1.1. Introductory example

This paper concerns the stability of non-linear normal modes in two degree of freedom systems. Instabilities in such cases are due to autoparametric excitation [1], that is, parametric excitation which is caused by the system itself, rather than by an external periodic driver. The investigation of stability involves the solution of a system of linear differential equations with periodic coefficients (Floquet theory). The typical behavior of such a system involves tongues of instability representing parametric resonances (Mathieu's equation, for example). Coexistence phenomenon refers to the circumstance in which some of these tongues of instability have closed up and disappeared. Their absence cloaks hidden instabilities which may emerge due to small changes in the system. This effect is important because it occurs in various mechanical systems.

We begin by illustrating the phenomenon with a physical example. This example, called "the particle

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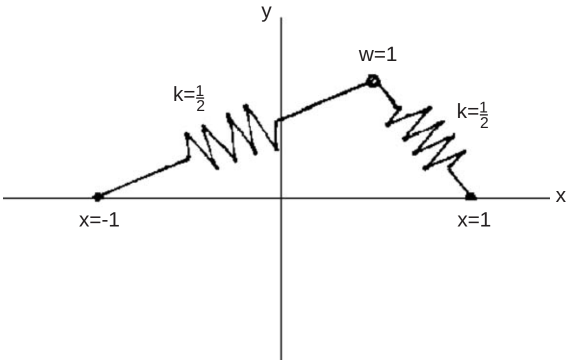


Fig. 1. The particle in the plane.

in the plane” by Yang and Rosenberg [2,3] who first studied it, involves a unit mass which is constrained to move in the $x$–$y$ plane, and is restrained by two linear springs, each with spring constant of $\frac{1}{2}$. The anchor points of the two springs are located on the $x$-axis at $x=1$ and $x=-1$. Each of the two springs has unstretched length $L$. See Fig. 1.

This autonomous two degree of freedom system has the following equations of motion [2]:

$$\ddot{x} + (x+1)f_1(x,y) + (x-1)f_2(x,y) = 0, \tag{1}$$

$$\ddot{y} + yf_1(x,y) + yf_2(x,y) = 0, \tag{2}$$

where

$$f_1(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1+x)^2 + y^2}}\right). \tag{3}$$

$$f_2(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1-x)^2 + y^2}}\right). \tag{4}$$

This system exhibits an exact solution corresponding to a mode of vibration in which the particle moves along the $x$-axis (the $x$-mode):

$$x = A\cos t, \quad y = 0. \tag{5}$$

In order to determine the stability of this motion, one must substitute $x = A\cos t + u,\ y = 0 + v$ into the equations of motion (1), (2) where $u$ and $v$ are small deviations from the motion (5), and then linearize in $u$ and $v$. The result is two linear differential equations on

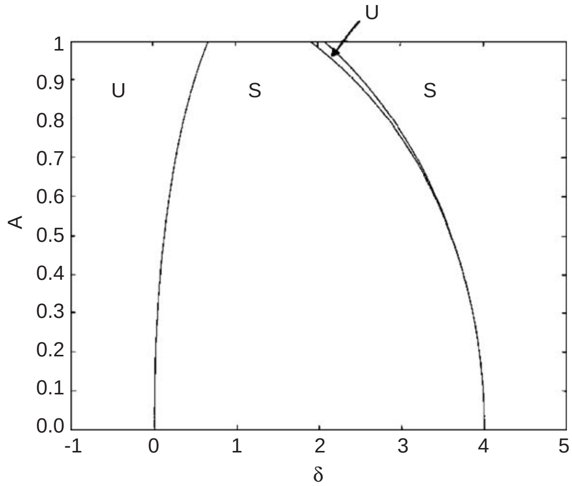


Fig. 2. Stability chart for Eq. (6). S = stable, U = unstable. Curves obtained by perturbation analysis.

$u$ and $v$. The $u$ equation turns out to be the simple harmonic oscillator, and cannot produce instability. The $v$ equation is

$$\frac{d^2v}{dt^2} + \left(\frac{\delta - A^2\cos^2 t}{1 - A^2\cos^2 t}\right)v = 0, \tag{6}$$

where $\delta = 1 - L$. For a particular pair of parameters $(A, \delta)$, Eq. (6) is said to be stable if all solutions to (6) are bounded, and unstable if an unbounded solution exists. A stability chart for Eq. (6) may be obtained by using either perturbation theory or numerical integration together with Floquet theory (see [4] for example). See Fig. 2. Note that although this equation exhibits an infinite number of tongues of instability, only one of them (emanating from the point $\delta=4$, $A=0$) is displayed, for convenience. (The tongues of instability emanate from $\delta = 4N^2$, $A=0$ for $N = 1, 2, 3, \ldots$, and becomes progressively narrower for increasing $N$.) Since the unstretched spring length $L > 0$, the parameter $\delta = 1 - L < 1$. Thus, the only tongue of instability for Eq. (6) which has physical significance is the one which emanates from $\delta = 0$, see Fig. 2.

Now we wish to compare the behavior of this system with a slightly perturbed system in which some extra stiffness is added. We add a spring which gives a force $-\Gamma y$ in the $y$-direction. This adds a term $+\Gamma y$ to the left-hand side of Eq. (2). The new system still exhibits the periodic solution (5), and its stability turns out to

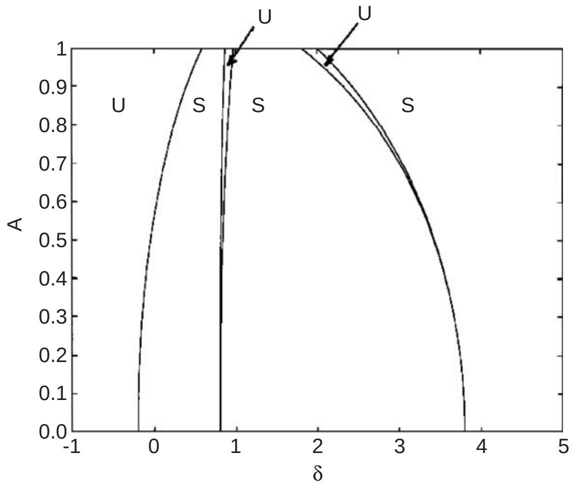


Fig. 3. Stability chart for Eq. (7) for $\Gamma = 0.2$. S = stable, U = unstable. Note the presence of an additional tongue of instability compared to Fig. 2. See text.

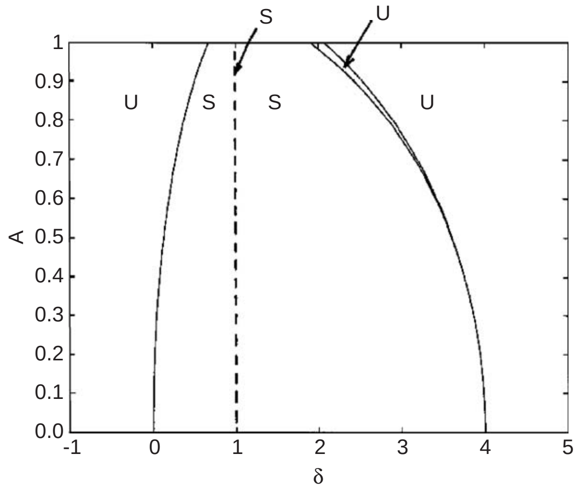


Fig. 4. Stability chart for Eq. (6) showing coexistence curve as a dashed line (here $\delta = 1$). Note that although the coexistence curve is itself stable, it may give rise to a tongue of instability if the system is perturbed.

be governed by the ODE.

$$\frac{d^2 v}{dt^2} + \left( \frac{\delta + \Gamma - (1+\Gamma) A^2 \cos^2 t}{1 - A^2 \cos^2 t} \right) v = 0. \tag{7}$$

Note that Eq. (7) reduces to (6) for $\Gamma = 0$. Fig. 3 shows the stability chart for Eq. (7).

Comparison of Figs. 2 and 3 shows that a new region of instability has occurred due to the small change made in the system. If an engineering design was based on Fig. 2, and if the actual engineering system involved slight departures from the model of Eq. (6), the appearance of such an unexpected region of instability could cause disastrous consequences. In this paper we investigate the possibility of the occurrence of such hidden instabilities in a class of two degree of freedom systems.

### 1.2. Coexistence phenomenon

The appearance of an unexpected instability region in the foregoing example may be explained by stating that Eq. (6) had buried in it an instability region of zero thickness [5]. This is shown in Fig. 4, which is a replot of Fig. 2 with the zero-thickness instability region displayed as a dashed line. This curve, which happens to have the simple equation $\delta = 1$, is characterized by the *coexistence* of two linearly independent periodic solutions of period $2\pi$. This condition is singular and so we are not surprised to find that nearly any perturbation of the original system (6), such as the reassignment of spring stiffnesses in (7), will produce an opening up of the zero-thickness instability region.

It should be mentioned that there are various other physical systems which are known to exhibit coexistence. These include a simplified model of a vibrating elastica [6], the elastic pendulum [4], rain–wind-induced vibrations [7], Josephson junctions [8] and coupled non-linear oscillators [9].

Coexistence phenomenon has been treated from a theoretical point of view in [10], and more recently in [4,11]. In this paper we use perturbation methods to rederive and extend the results given in [4,10,11]. In particular, we address the question of finding conditions under which a class of linear ODEs with periodic coefficients will exhibit coexistence phenomenon.

## 2. Motivating application

We wish to study autoparametric excitation in a class of systems which on the one hand have the following very general expressions for kinetic energy $T$

and potential energy $V$:

$$T = \beta_1(x,y)\dot{x}^2 + \beta_2(x,y)\dot{x}\dot{y} + \beta_3(x,y)\dot{y}^2, \tag{8}$$

$$V = \tfrac{1}{2}\,\omega_1^2 x^2 + \tfrac{1}{2}\,\omega_2^2 y^2 + \alpha_{40}x^4 + \alpha_{31}x^3 y + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4 \tag{9}$$

and on the other hand generalize the particle in the plane example by exhibiting an $x$-mode of the form of Eq. (5):

$$x = A\cos t, \quad y = 0. \tag{10}$$

Writing Lagrange's equations for the system (8), (9), we find that in order for (10) to be a solution, we must have $\alpha_{40}=0$, $\alpha_{31}=0$, $\beta_2=0$ and $\beta_1=\omega_1^2/2$. Choosing $\omega_1=1$ without loss of generality, we obtain the following expressions for $T$ and $V$:

$$T = \tfrac{1}{2}\,\dot{x}^2 + \beta_3(x,y)\dot{y}^2, \tag{11}$$

$$V = \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,\omega_2^2 y^2 + \alpha_{22}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4. \tag{12}$$

We further assume that $\beta_3(x,y)$ has the following form:

$$\beta_3(x,y) = \beta_{00} + \beta_{01}x + \beta_{10}y + \beta_{02}x^2 + \beta_{11}xy + \beta_{20}y^2. \tag{13}$$

Now we investigate the linear stability of the $x$-mode (10). We set $x = A\cos t + u,\;\; y = 0 + v$ in Lagrange's equations and then linearize in $u$ and $v$. This gives the $u$ equation as $\ddot{u}+u=0$ and the $v$ equation as

$$\begin{aligned}&(2\beta_{00} + A^2\beta_{02} + 2A\beta_{01}\cos t + A^2\beta_{02}\cos 2t)\ddot{v}\\ &\quad + (-2A\beta_{01}\sin t - 2A^2\beta_{02}\sin 2t)\dot{v}\\ &\quad + (\omega_2^2 + A^2\alpha_{22} + A^2\alpha_{22}\cos 2t)v = 0.\end{aligned} \tag{14}$$

This leads us to consider the following abbreviated form of (14):

$$\begin{aligned}&(1 + a_1\cos t + a_2\cos 2t)\ddot{v}\\ &\quad + (b_1\sin t + b_2\sin 2t)\dot{v}\\ &\quad + (\delta + c_1\cos t + c_2\cos 2t)v = 0,\end{aligned} \tag{15}$$

where

$$a_1 = \frac{2A\beta_{01}}{2\beta_{00} + A^2\beta_{02}},$$

$$a_2 = \frac{A^2\beta_{02}}{2\beta_{00} + A^2\beta_{02}},$$

$$b_1 = \frac{-2A\beta_{01}}{2\beta_{00} + A^2\beta_{02}} = -a_1,$$

$$b_2 = \frac{-2A^2\beta_{02}}{2\beta_{00} + A^2\beta_{02}} = -2a_2,$$

$$\delta = \frac{\omega_2^2 + A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}},$$

$$c_1 = 0,$$

$$c_2 = \frac{A^2\alpha_{22}}{2\beta_{00} + A^2\beta_{02}}. \tag{16}$$

## 3. Generalized Ince's equation

We come now to the main content of this paper, namely a study of the coexistence phenomenon in the ODE (15):

$$\begin{aligned}&(1 + a_1\cos t + a_2\cos 2t)\ddot{v}\\ &\quad + (b_1\sin t + b_2\sin 2t)\dot{v}\\ &\quad + (\delta + c_1\cos t + c_2\cos 2t)v = 0.\end{aligned} \tag{17}$$

In the case that $a_2=0$, $b_2=0$ and $c_2=0$, Eq. (17) reduces to a well-known ODE called Ince's equation. Coexistence in Ince's equation has been studied in [4,10,11]. In the rest of this paper, we generalize the previously obtained results for Ince's equation to apply to the generalized Ince's equation (17).

Eq. (17) is a linear ODE with periodic coefficients having period $2\pi$. From Floquet theory we know that the transition curves separating regions of stability from regions of instability are defined by sets of parameter values that allow periodic solutions of period $2\pi$ or $4\pi$. These curves can be found by using the method of harmonic balance. Periodicity enables the solution to be written in the form of a Fourier series:

$$v(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{nt}{2} + \sum_{n=1}^{\infty} B_n\sin\frac{nt}{2}. \tag{18}$$

Substituting (18) into (17) and trigonometrically reducing and collecting terms gives an infinite set of coupled equations. These uncouple into four sets of equations on even and odd cosine ($A_n$) and sine ($B_n$) coefficients. For example, the $A$-even coefficients satisfy the following equations:

*A*-even

$$\begin{bmatrix} \delta & -\frac{1}{2}a_1-\frac{1}{2}b_1+\frac{1}{2}c_1 & -2a_2-b_2+\frac{1}{2}c_2 & 0 & \cdots \\ c_1 & \delta-1 & -\frac{1}{2}a_2-\frac{1}{2}b_2+\frac{1}{2}c_2 & -2a_1-b_1+\frac{1}{2}c_1 & \cdots \\ c_2 & -\frac{1}{2}a_1+\frac{1}{2}b_1+\frac{1}{2}c_1 & \delta-4 & -\frac{9}{2}a_1-\frac{3}{2}b_1+\frac{1}{2}c_1 & \cdots \\ 0 & -\frac{1}{2}a_2+\frac{1}{2}b_2+\frac{1}{2}c_2 & -2a_1+b_1+\frac{1}{2}c_1 & \delta-9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ \vdots \end{bmatrix}=0.$$

To simplify the notation, we introduce the following substitutions:

$$T(n)=\delta-\left(\frac{n}{2}\right)^2,$$

$$M(n)=\frac{1}{2}\left(-\left(\frac{n}{2}\right)^2a_1+\frac{n}{2}b_1+c_1\right),$$

$$P(n)=\frac{1}{2}\left(-\left(\frac{n}{2}\right)^2a_2+\frac{n}{2}b_2+c_2\right). \tag{19}$$

The four sets of penta-diagonal matrix equations may then be written as

*A*-even

$$\begin{bmatrix} T(0) & M(-2) & P(-4) & 0 & 0 & 0 & \cdots \\ 2M(0) & T(2)+P(-2) & M(-4) & P(-6) & 0 & 0 & \cdots \\ 2P(0) & M(2) & T(4) & M(-6) & P(-8) & 0 & \cdots \\ 0 & P(2) & M(4) & T(6) & M(-8) & P(-10) & \cdots \\ 0 & 0 & P(4) & M(6) & T(8) & M(-10) & \cdots \\ 0 & 0 & 0 & P(6) & M(8) & T(10) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} A_0 \\ A_2 \\ A_4 \\ A_6 \\ A_8 \\ A_{10} \\ \vdots \end{bmatrix}=0,$$

*B*-even

$$\begin{bmatrix} T(2)-P(-2) & M(-4) & P(-6) & 0 & 0 & \cdots \\ M(2) & T(4) & M(-6) & P(-8) & 0 & \cdots \\ P(2) & M(4) & T(6) & M(-8) & P(-10) & \cdots \\ 0 & P(4) & M(6) & T(8) & M(-10) & \cdots \\ 0 & 0 & P(6) & M(8) & T(10) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{10} \\ \vdots \end{bmatrix}=0,$$

*A*-odd

$$\begin{bmatrix} T(1)+M(-1) & M(-3)+P(-3) & P(-5) & 0 & 0 & \cdots \\ M(1)+P(-1) & T(3) & M(-5) & P(-7) & 0 & \cdots \\ P(1) & M(3) & T(5) & M(-7) & P(-9) & \cdots \\ 0 & P(3) & M(5) & T(7) & M(-9) & \cdots \\ 0 & 0 & P(5) & M(7) & T(9) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \\ A_7 \\ A_9 \\ \vdots \end{bmatrix} = 0,$$

*B*-odd

$$\begin{bmatrix} T(1)-M(-1) & M(-3)-P(-3) & P(-5) & 0 & 0 & \cdots \\ M(1)-P(-1) & T(3) & M(-5) & P(-7) & 0 & \cdots \\ P(1) & M(3) & T(5) & M(-7) & P(-9) & \cdots \\ 0 & P(3) & M(5) & T(7) & M(-9) & \cdots \\ 0 & 0 & P(5) & M(7) & T(9) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ B_9 \\ \vdots \end{bmatrix} = 0.$$

Each of the four above sets of equations is homogeneous and of infinite order, so for a non-trivial solution the determinants must vanish. Note that the resulting determinants for *A*-odd and *B*-odd are identical except for the first row and the first column. A comparable similarity exists between the determinants for *A*-even and *B*-even. Although generally the vanishing of, say, the *A*-odd determinant will give a completely different result than that of the *B*-odd determinant, nevertheless there may exist a special relationship between the coefficients such that the two results will give infinitely many identical branches, that is, infinitely many of the transition curves will be identical, in which case the associated instability regions will disappear (or rather will have zero width). On such transition curves we will have both an odd and an even periodic motion, that is, two linearly independent periodic motions will **coexist**. In order to derive conditions for coexistence, we write any one of the above infinite penta-diagonal determinants in the form

$$\begin{vmatrix} R & R & R & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ R & R & R & R & 0 & 0 & 0 & 0 & 0 & \cdots \\ R & R & R & R & Y & 0 & 0 & 0 & 0 & \cdots \\ 0 & R & R & R & Y & Y & 0 & 0 & 0 & \cdots \\ 0 & 0 & X & X & S & S & S & 0 & 0 & \cdots \\ 0 & 0 & 0 & X & S & S & S & S & 0 & \cdots \\ 0 & 0 & 0 & 0 & S & S & S & S & S & \cdots \\ 0 & 0 & 0 & 0 & 0 & S & S & S & S & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & S & S & S & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0. \quad (20)$$

If all three of the *X* terms vanish, or if all three of the *Y* terms vanish, the determinant will decompose into two determinants, one involving only the *R* terms, and the other involving only the *S* terms. Since the *A*-odd and *B*-odd determinants are identical except for the upper left-hand corner, the corresponding determinant of (20) involving only the *S* terms will be the same for both *A*-odd and *B*-odd, and we will have coexistence. The vanishing of the three *X* terms or of the three *Y* terms turns out to give the following conditions:

$$P(n-2)=0, \quad M(n)=0, \quad P(n)=0, \qquad (21)$$

where $n$ can be any integer,

$$n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \; .$$

From our definitions (19) of $M$ and $P$, we are left with the following conditions for coexistence in the generalized Ince's equation (17):

$$c_1 = \left(\frac{n}{2}\right)^2 a_1 - \frac{n}{2} b_1,$$

$$b_2 = (n-1)a_2,$$

$$c_2 = \left(\frac{n}{2}\right)^2 a_2 - \frac{n}{2} b_2. \qquad (22)$$

Thus, coexistence will occur in the generalized Ince equation (17) if Eqs. (22) hold for any integer $n$, positive, negative or zero.

Note that in the special case $a_2 = b_2 = c_2 = 0$, Eq. (17) becomes Ince's equation

$$(1 + a_1 \cos t)\ddot{v} + (b_1 \sin t)\dot{v} + (\delta + c_1 \cos t)v = 0. \tag{23}$$

In this case the matrices become tri-diagonal (instead of penta-diagonal) and the condition for coexistence reduces to just a single equation [4,10]

$$c_1 = \left(\frac{n}{2}\right)^2 a_1 - \frac{n}{2} b_1. \tag{24}$$

Note also that in the parallel case $a_1 = b_1 = c_1 = 0$, Eq. (17) again becomes a version of Ince's equation:

$$(1 + a_2 \cos 2t)\ddot{v} + (b_2 \sin 2t)\dot{v} + (\delta + c_2 \cos 2t)v = 0. \tag{25}$$

In this case we set $\tau = 2t$ giving

$$(1 + a_2 \cos\tau)\ddot{v} + \left(\frac{b_2}{2}\sin\tau\right)\dot{v} + \left(\delta^* + \frac{c_2}{4}\cos\tau\right)v = 0, \tag{26}$$

which is of the form of Eq. (23) with $a_1 = a_2$, $b_1 = b_2/2$, $c_1 = c_2/4$ and $\delta^* = \delta/4$, whereupon the condition (24) for coexistence becomes:

$$c_2 = n^2 a_2 - n b_2. \tag{27}$$

In related work, it has been shown [12] that even more complicated versions of Ince's equation cannot support coexistence. For example, the equation

$$(1 + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t)\ddot{v} + (b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t)\dot{v} + (\delta + c_1 \cos t + c_2 \cos 2t + c_3 \cos 3t)v = 0 \tag{28}$$

gives rise to four 7-diagonal determinants and requires six conditions to be met in order for coexistence to occur (cf. Eqs. (21)). These conditions turn out to be self-contradictory, so Eq. (28) cannot support coexistence (unless some of the coefficients are zero, thereby reducing it to the form of Eq. (17)).

Note that the coexistence conditions (22) do not involve the parameter $\delta$ in Eq. (17). Once the parameters of the system have been chosen to satisfy the coexistence conditions (22), the vanishing of the associated determinant (20) will relate $\delta$ to the other parameters of the system.

## 4. Application to stability of motion

Earlier in this paper we showed that the stability of the $x$-mode, Eq. (10), in system (11), (12), (13) was governed by the generalized Ince's equation (17) with coefficients given by Eq. (16). From (16) we substitute $c_1 = 0$ and $b_1 = -a_1$ into the first of the coexistence conditions (22) with the result

$$0 = \left(\frac{n}{2}\right)^2 a_1 - \frac{n}{2}(-a_1), \tag{29}$$

which is satisfied by either $n = -2$ or $n = 0$ or $a_1 = 0$.

Next, from (16) we substitute $b_2 = -2a_2$ into the second of the coexistence conditions (22) with the result

$$-2a_2 = (n-1)a_2, \tag{30}$$

which is satisfied by either $n = -1$ or $a_2 = 0$.

Thus, we see that if both $a_1$ and $a_2$ are non-zero, then coexistence cannot occur in the general system defined by Eqs. (11), (12), (13), since there is no integer $n$ which can satisfy the conditions (22). From the definitions (16) of $a_1$ and $a_2$, this assumes that both $\beta_{01}$ and $\beta_{02}$ are non-zero (assuming $A > 0$). (Recall that the $\beta_{ij}$ coefficients occur in the kinetic energy $T$, see Eqs. (11), (13).)

Note that if $\beta_{01} = 0$ but $\beta_{02}$ does not vanish, then coexistence is possible. However, in this case Eq. (17) reduces to Ince's equation, which is well-known to support coexistence [4,10].

## 5. Another application

In this Section we extend the foregoing work by considering systems in which the $x$-mode satisfies the non-linear ODE:

$$\ddot{x} + x + x^3 = 0, \tag{31}$$

which has a solution in terms of the Jacobian elliptic function cn

$$x = A\,\mathrm{cn}(\alpha t, k), \tag{32}$$

where [13, p. 80]

$$\alpha = \sqrt{A^2 + 1}, \quad k = \frac{A}{\sqrt{2(A^2+1)}}. \tag{33}$$

This requires that we relax the condition that $\alpha_{40}=0$ (cf. Eqs. (9) and (12)), and we take:

$$T=\tfrac{1}{2}\dot{x}^2+\beta_3(x,y)\dot{y}^2, \tag{34}$$

$$V=\tfrac{1}{2}x^2+\tfrac{1}{4}x^4+\tfrac{1}{2}\omega_2^2y^2+\alpha_{22}x^2y^2+\alpha_{13}xy^3+\alpha_{04}y^4, \tag{35}$$

$$\beta_3(x,y)=\beta_{00}+\beta_{01}x+\beta_{10}y+\beta_{02}x^2+\beta_{11}xy+\beta_{20}y^2. \tag{36}$$

We set $x=A\,\mathrm{cn}(\alpha t,k)+u,\ \ y=0+v$ in Lagrange's equations and then linearize in $u$ and $v$. This gives the $v$ equation as

$$2(\beta_{02}A^2\,\mathrm{cn}^2(\alpha t,k)+\beta_{01}A\,\mathrm{cn}(\alpha t,k)+\beta_{00})\ddot{v}-\alpha\mathrm{dn}(\alpha t,k)\mathrm{sn}(\alpha t,k)(2\beta_{01}A+4\beta_{02}A^2\mathrm{cn}(\alpha t,k))\dot{v}+(2\alpha_{22}A^2\,\mathrm{cn}^2(\alpha t,k)+\omega_2^2)v=0. \tag{37}$$

Although Eq. (37) has coefficients involving Jacobian elliptic functions, we may transform it to a generalized Ince equation by utilizing a transformation given in [10]. We begin by replacing $t$ with a new time variable $T=\alpha t$, so that $\mathrm{cn}(\alpha t,k)=\mathrm{cn}(T,k)$. Then we replace $T$ by $\tau$, where

$$\mathrm{d}T=\frac{\mathrm{d}\tau}{\sqrt{1-k^2\sin^2\tau}}. \tag{38}$$

This turns out to convert the Jacobian elliptic functions to trig functions [14] as follows:

$$\mathrm{sn}(T,k)=\sin\tau,$$

$$\mathrm{cn}(T,k)=\cos\tau,$$

$$\mathrm{dn}(T,k)=\sqrt{1-k^2\sin^2\tau}. \tag{39}$$

The result of these transformations is to replace Eq. (37) by the following generalized Ince equation:

$$(1+a_1\cos\tau+a_2\cos 2\tau+a_3\cos 3\tau+a_4\cos 4\tau)v''+(b_1\sin\tau+b_2\sin 2\tau+b_3\sin 3\tau+b_4\sin 4\tau)v'+(\delta+c_1\cos\tau+c_2\cos 2\tau+c_3\cos 3\tau+c_4\cos 4\tau)v=0, \tag{40}$$

where the coefficients $a_i$, $b_i$ and $c_i$ are given as follows:

$$a_1=\frac{2A\beta_{01}(1-\frac{1}{4}k^2)}{a_0}, \tag{41}$$

$$a_2=\frac{\beta_{00}k^2+\beta_{02}A^2}{a_0}, \tag{42}$$

$$a_3=\frac{\frac{1}{2}\beta_{01}Ak^2}{a_0}, \tag{43}$$

$$a_4=\frac{\frac{1}{4}A^2k^2\beta_{02}}{a_0}, \tag{44}$$

$$b_1=\frac{-\beta_{01}A(2-k^2)}{a_0}, \tag{45}$$

$$b_2=\frac{-2\beta_{02}A^2(1-\frac{1}{4}k^2)-\beta_{00}k^2}{a_0}, \tag{46}$$

$$b_3=\frac{-\beta_{01}Ak^2}{a_0}, \tag{47}$$

$$b_4=\frac{-\frac{3}{4}A^2k^2\beta_{02}}{a_0}, \tag{48}$$

$$\delta=\frac{\omega_2^2+\alpha_{22}A^2}{a_0\alpha^2}, \tag{49}$$

$$c_1=0, \tag{50}$$

$$c_2=\frac{\alpha_{22}A^2}{a_0\alpha^2}, \tag{51}$$

$$c_3=0, \tag{52}$$

$$c_4=0, \tag{53}$$

where

$$a_0=\beta_{00}(2-k^2)+\beta_{02}A^2(1-\tfrac{1}{4}k^2). \tag{54}$$

As mentioned in connection with Eq. (28) above, Eq. (40) cannot in general support coexistence. However, if $\beta_{01}=0$, the trigonometric terms in Eq. (40) with arguments of $\tau$ and $3\tau$ will vanish, leaving an equation which can easily be transformed into the generalized Ince equation (17) by replacing $\tau$ by $z=2\tau$. Once this transformation is completed, conditions for coexistence in the resulting equation will be given by Eqs. (22). Carrying out this plan yields three equations corresponding to Eqs. (22). The equation which corresponds to the second of Eqs. (22) turns out to be

$$(n+1/2)\alpha^2\beta_{02}A^2k^2=0, \tag{55}$$

which requires that $n=-1/2$ and thus cannot be satisfied by any integer value of $n$. However, Eq. (55) as well as the other two equations coming from Eqs. (22) can be satisfied by taking $\beta_{02} = 0$.

So we conclude that in order for coexistence to occur in Eq. (37), both $\beta_{01}$ and $\beta_{02}$ must be taken equal to zero. This simplifies Eq. (40) to the following:

$$(1 + a_2 \cos 2\tau)v'' + (b_2 \sin 2\tau)v' + (\delta + c_2 \cos 2\tau)v = 0. \tag{56}$$

This is of the form of Eq. (25) and as was discussed above, involves a single condition (27) for coexistence

$$c_2 = n^2 a_2 - n b_2. \tag{57}$$

Using Eqs. (41)–(53), Eq. (57) becomes

$$(-\alpha^2 \beta_{00} k^2) n^2 + (-\alpha^2 \beta_{00} k^2) n + \alpha_{22} A^2 = 0, \tag{58}$$

which becomes simplified by using Eqs. (33):

$$n^2 + n - \frac{2\alpha_{22}}{\beta_{00}} = 0. \tag{59}$$

The condition for coexistence therefore becomes simply

$$\frac{\alpha_{22}}{\beta_{00}} = \frac{n(n+1)}{2}, \tag{60}$$

where $n$ is an integer, positive, negative or zero.

## 6. Example

As an example, we may take $\beta_{00}=1/2$ and $\alpha_{22}=1/2$, which from Eq. (60) corresponds to $n=1$ and $n=-2$. Eqs. (34), (35) become

$$T = \tfrac{1}{2}\dot{x}^2 + (\tfrac{1}{2} + \beta_{10}x + \beta_{20}x^2 + \beta_{11}xy)\dot{y}^2, \tag{61}$$

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + \tfrac{1}{2}\omega_2^2 y^2 + \tfrac{1}{2}x^2y^2 + \alpha_{13}xy^3 + \alpha_{04}y^4. \tag{62}$$

In order to consider the simplest possible such example, we take $\beta_{10}=\beta_{20}=\beta_{11}=\alpha_{13}=\alpha_{04}=0$, for which case Lagrange's equations become

$$\ddot{x} + x + x^3 + xy^2 = 0, \tag{63}$$

$$\ddot{y} + \omega_2^2 y + x^2 y = 0. \tag{64}$$

This system exhibits the exact solution (the $x$-mode):

$$x = A\,\mathrm{cn}(\alpha t, k), \quad y = 0, \tag{65}$$

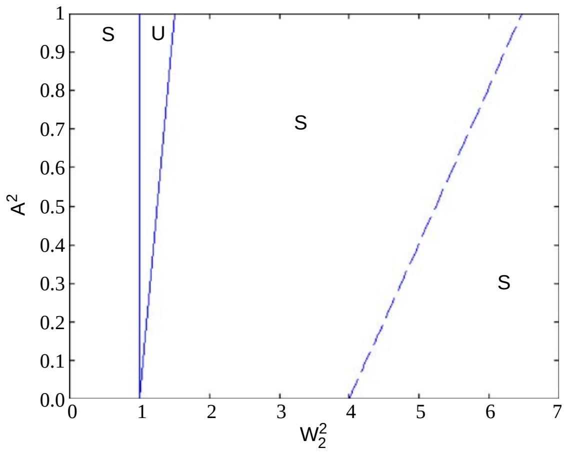


Fig. 5. Stability chart for Eq. (66). S = stable, U = unstable. Curves obtained by perturbation analysis. The dashed line is a coexistence curve, which is stable.

where $\alpha$ and $k$ are given by Eq. (33). The stability of the $x$-mode depends upon the two parameters $\omega_2$ and $A$, and is governed by the ODE (37), which becomes

$$\ddot{v} + (\omega_2^2 + A^2\,\mathrm{cn}^2(\alpha t, k))v = 0. \tag{66}$$

The stability chart corresponding to Eq. (66) consists of transition curves which may be displayed in the $\omega_2^2$–$A^2$ plane. Since the period of the variable coefficient $\mathrm{cn}^2(\alpha t, k)$ approaches $\pi$ as $A$ approaches zero, we may expect instability tongues to emanate from the $\omega_2^2$ axis at each of the points $\omega_2^2 = n^2$, where $n = 1, 2, 3, \ldots$. However, because $\alpha_{22}$ and $\beta_{00}$ have been chosen to satisfy the coexistence condition (60) for $n=1$ and $-2$, there are no even tongues and only one odd tongue, which emanates from the point $\omega_2^2=1$, $A^2 = 0$ [4]. See Fig. 5, which shows this single instability tongue as well as a coexistence curve emanating from $\omega_2^2 = 4$, $A^2 = 0$. Fig. 5 was obtained as follows:

Eq. (66) is a version of Lame's equation [15]. Following the procedure given in Eqs. (38), (39), it can be transformed to

$$(3A^2 + 4 + A^2 \cos 2\tau)v'' - A^2 \sin 2\tau v' + (4\omega_2^2 + 2A^2 + 2A^2 \cos 2\tau)v = 0. \tag{67}$$

Note that Eq. (67) has the exact solution $v = \cos\tau$ corresponding to the parameter $\omega_2^2 = 1$. Therefore, the straight line $\omega_2^2 = 1$ is a transition curve as shown in Fig. 5. Similarly, Eq. (67) has the exact solution

$v = \sin\tau$ corresponding to the parameter $\omega_2^2 = 1 + A^2/2$, which also plots as a straight line in Fig. 5.

In order to obtain an expression for the coexistence curves, we may use a regular perturbation method [16]. We expand

$$\omega_2^2 = n^2 + k_1 A^2 + k_2 A^4 + \cdots, \tag{68}$$

$$v = \left\{ \begin{matrix} \sin n\tau \\ \cos n\tau \end{matrix} \right\} + v_1 A^2 + v_2 A^4 + \cdots. \tag{69}$$

We substitute Eqs. (68), (69) into Eq. (67), collect terms, and choose the values of the coefficients $k_i$ to eliminate secular terms at each order of $A^2$, as usual in regular perturbations [16]. Doing this for $n = 2$ we obtain the same result for both sin and cos choices in Eq. (69), signifying coexistence. The resulting curve is displayed in Fig. 5 and has the equation (obtained by using macsyma to do the computer algebra):

$$\omega_2^2 = 4 + \frac{5A^2}{2} - \frac{5A^4}{96} + \frac{5A^6}{128} - \frac{26665A^8}{884736} + \frac{9385A^{10}}{393216} - \frac{19720235A^{12}}{1019215872} + \cdots. \tag{70}$$

## 7. Conclusions

We have obtained conditions (22) for coexistence to occur in the generalized Ince equation (17). These conditions are more numerous and thus more difficult to meet than the comparable condition for Ince's equation:

$$(1 + a_1 \cos t)\ddot{v} + b_1 \sin t\, \dot{v} + (\delta + c_1 \cos t)v = 0. \tag{71}$$

The necessary and sufficient condition for coexistence to occur in (71) has been obtained in [10] and can be written in the form

$$M(n) = \frac{1}{2}\left(-\left(\frac{n}{2}\right)^2 a_1 + \frac{n}{2} b_1 + c_1\right) = 0, \tag{72}$$

where $n$ can be any integer,

$$n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots.$$

That is, coexistence will occur in (71) iff condition (72) is satisfied for any integer value of $n$.

In applications to the stability of the $x$-mode in the class of two degree of freedom systems (8), (9) considered in this paper, we have shown that in general coexistence will not occur if the system is sufficiently complicated, i.e., if both of the coefficients $\beta_{01}$ and $\beta_{02}$ occurring in Eq. (13) are non-zero. The reason for this is that the equation governing stability is the generalized Ince's equation (17), and the conditions for coexistence to occur in this equation are more difficult to meet than for Ince's equation (71).

We have also shown that the same general procedure can be used on problems in which the $x$-mode satisfies a non-linear ODE, Eq. (31).

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