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ANALYTICAL APPROXIMATION FOR PERIOD-DOUBLING FOLLOWING A HOPF BIFURCATION

Richard H. Rand
Department of Theoretical & Applied Mechanics
Cornell University, Ithaca, NY 14853

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Introduction

This work is concerned with the behavior of a system of n autonomous ordinary differential equations of the form $\dot{x} = F(x,\mu)$, where x and $F(x,\mu)$ are n-vectors, and where μ is a scalar parameter. It frequently happens that a change in μ causes an equilibrium point to change its stability resulting in the birth of a periodic motion called a limit cycle, a process known as a Hopf bifurcation [1]. If the change in μ is then continued, the limit cycle may itself undergo a change of form via a period-doubling bifurcation [2]. This scenario is often the first step in a sequence of period-doublings leading to chaos.

The goal of this work is to suggest an analytic approach which will yield an approximation for μ^*, the critical value of μ corresponding to the first period-doubling bifurcation. The idea of the method is to use center manifold theory [3,4] to approximate the newly-born limit cycle, and then to use that approximation to investigate the stability of the limit cycle. The critical value μ^* corresponds to the change of stability of the limit cycle (and to the disintegration of the center manifold).

Specifically, the approach suggested in this work consists of three steps and is aimed at predicting the parameter value associated with period-doubling:

1. a center manifold reduction,
2. a perturbation study of the limit cycle on the center manifold,
3. a problem in Floquet theory arising from the stability of the limit cycle in directions normal to the center manifold.

We begin with a 3-dimensional example and then generalize the approach to n dimensions.

An Example

In order to illustrate the phenomena and explain the method, we refer to the following example:

(1.1) $$\dot{x} = \mu x - y - x z$$

(1.2) $$\dot{y} = \mu y + x$$

(1.3) $$\dot{z} = - z + x^2 z + y^2$$

This example may be thought of as a feedback control system consisting of a damped linear oscillator in x,y variables and a control variable z. The nonlinear terms represent feedback or coupling. The origin $x = y = z = 0$ is an equilibrium, and linearization about the origin shows it to be stable for $\mu < 0$ and unstable for $\mu > 0$. The resulting Hopf bifurcation occurs at $\mu = 0$ in a center manifold which is tangent to the x-y plane at the origin. As may be seen from the Figures, the limit cycle occurs for $\mu > 0$, is initially stable, and period-doubles at about $\mu = \mu^* = 0.45$, and again at about $\mu = 0.48$.

STEP 1

In order to approximate the center manifold, we set

(2) $$z = a x^2 + b x y + c y^2 + O(3)$$

Differentiating (2) and using eqs.(1) gives:

$$\dot{z} = 2a\, x\, \dot{x} + b\, \dot{x}\, y + b\, x\, \dot{y} + 2c\, y\, \dot{y} + O(3)$$

$$-z+x^2z+y^2 = (2ax+by)\ (\mu x-y-xz) + (bx+2cy)\ (\mu y+x) + O(3)$$

$$-ax^2-bxy-cy^2+y^2=(2ax+by)(\mu x-y) + (bx+2cy)\ (\mu y+x) + O(3)$$

Equating to zero the coefficients of x^2,xy and y^2 gives three linear equations on a,b and c:

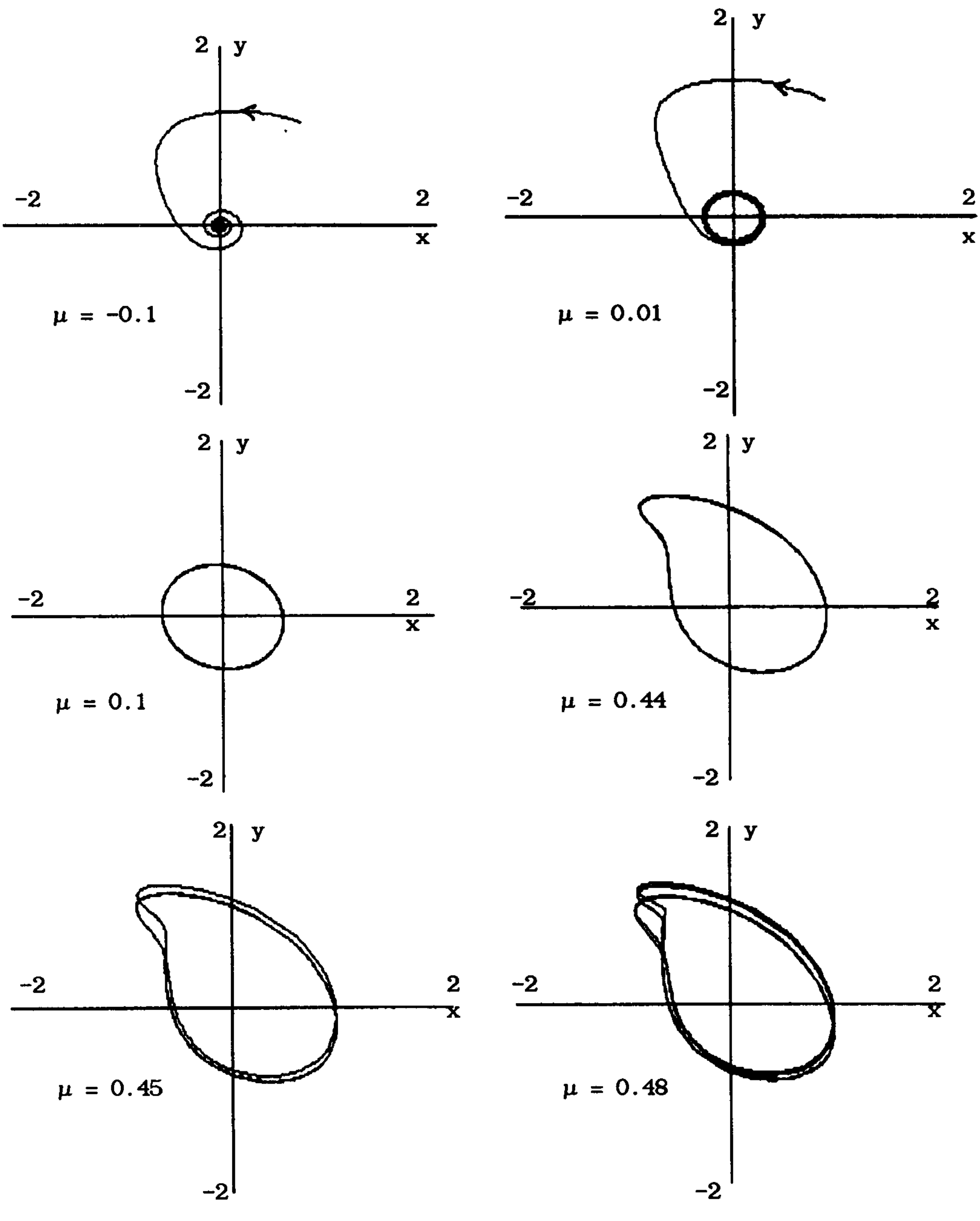
$$x^2:\ -a = 2a\mu + b$$

$$xy:\ -b = 2\mu b - 2a + 2c$$

$$y^2:\ -c + 1 = -b + 2\mu c$$

Solving for a,b,c and expanding the results for small μ gives:

(3) $$a = \frac{2}{5} - \frac{28}{25}\mu, \quad b = -\frac{2}{5} + \frac{8}{25}\mu, \quad c = \frac{3}{5} - \frac{22}{25}\mu$$



Dynamics of eqs.(1) obtained by Runge Kutta numerical integration and displayed by projection onto the xy plane for various values of μ. For $\mu = -0.1$ the equilibrium point at the origin is stable and no limit cycle exists. A Hopf bifurcation occurs at $\mu = 0$, resulting in the birth of a stable limit cycle (see $\mu = 0.01$), which grows in size (see $\mu = 0.1$) until at $\mu = 0.45$ it period-doubles. Period-doubling again occurs at $\mu = 0.48$. Note that the apparent self-intersections for $\mu = 0.45$ and $\mu = 0.48$ are vestiges of the projection.

STEP 2

Substituting the center manifold (2) into (1.1) and (1.2) gives the following approximate flow on the center manifold:

(4.1) $$\dot{x} = \mu x - y - a x^3 - b x^2 y - c xy^2$$

(4.2) $$\dot{y} = \mu y + x$$

where a,b,c are given by (3). An approximation for the limit cycle exhibited by eqs.(4) can be obtained by a variety of methods [4], such as Lindstedt's method or averaging. We obtain an equivalent result by the method of normal forms [4], application of which yields the near-identity transformation:

(5.1) $$x = u + \frac{81}{400} u^3 - \frac{3}{25} \mu u^3 - \frac{61}{800} u^2 v + \frac{43}{200} \mu u^2 v + \frac{141}{400} uv^2 - \frac{121}{400} \mu uv^2 - \frac{121}{800} v^3$$

(5.2) $$y = v - \frac{79}{800} u^3 + \frac{1}{2} \mu u^3 + \frac{101}{400} u^2 v + \frac{31}{400} \mu u^2 v - \frac{59}{800} uv^2 + \frac{121}{200} \mu uv^2 + \frac{121}{400} v^3$$

When eqs.(5) are applied to eqs.(4) we obtain the transformed eqs:

(6.1) $$\dot{u} = \mu u - v - \frac{9}{40} u(u^2+v^2) + \frac{1}{20} v(u^2+v^2)$$

(6.2) $$\dot{v} = \mu v + u - \frac{9}{40} v(u^2+v^2) - \frac{1}{20} u(u^2+v^2)$$

Eqs.(6) are next written in polar coordinates, $u = r \cos\theta$, $v = r \sin\theta$,

(7) $$\dot{r} = \mu r - \frac{9}{40} r^3, \quad \dot{\theta} = 1 - \frac{1}{20} r^2$$

From the first of eqs.(7) we obtain the approximate limit cycle amplitude

(8) $$r_{1c} = \frac{2\sqrt{10}}{3} \sqrt{\mu}$$

and the following approximation for the limit cycle:

(9) $$x_{1c} = \frac{2\sqrt{10}}{3} \sqrt{\mu} \cos\theta, \quad y_{1c} = \frac{2\sqrt{10}}{3} \sqrt{\mu} \sin\theta$$

in which $\theta = (1 - \frac{2}{9} \mu)t + \theta(0)$, from (7),(8) STEP 3

Now we return to eq.(1.3) which governs the stability of the limit cycle and of the center manifold which contains it. Note that the limit cycle cannot period-double as long as it lies in the center manifold, since the latter is two dimensional and trajectories cannot self-intersect, cf. the Figures. Substitution of (9) into (2) gives an expression for z on the limit cycle, call it z_{lc}. To investigate its stability, set in eq.(1.3)

(10) $$z = z_{lc} + \delta z$$

and linearize in the variation δz, giving

(11) $$\delta\dot{z} = (-1 + x_{lc}^2)\ \delta z$$

or using eq.(9),

(12) $$\delta\dot{z} = (-1 + \frac{20}{9}\mu + \frac{20}{9}\mu \cos 2\theta)\ \delta z$$

Eq.(12) has the general solution

$$\delta z(t) = \delta z(0) \exp \int_0^t (-1 + \frac{20}{9}\mu + \frac{20}{9}\mu \cos 2\theta)\ dt$$

The transition from stable to unstable occurs when $\delta z(T) = \delta z(0)$, where T is the period of the limit cycle oscillation. Thus μ^* is given by

$$\int_0^{2\pi} (-1 + \frac{20}{9}\mu^* + \frac{20}{9}\mu^* \cos 2\theta)\ d\theta = 0$$

(13) $$\mu^* = \frac{9}{20} = 0.45$$

which agrees with the results of numerical integration, see the Figures.

Generalization

Eqs.(1) may be viewed as a special case of the system

$$\dot{x} = \mu x - y + f(x,y,\bar{z},\mu) \tag{14.1}$$

$$\dot{y} = \mu y + x + g(x,y,\bar{z},\mu) \tag{14.2}$$

$$\dot{\bar{z}} = A(\mu)\ \bar{z} + \bar{h}(x,y,\bar{z},\mu) \tag{14.3}$$

in which $\bar{z} = [z_1,z_2,\ldots,z_{n-2}]$ is an (n-2) vector, $A(\mu)$ is an $(n-2)\times(n-2)$ matrix with all eigenvalues having negative real parts at $\mu = 0$, and f,g and $\bar{h} = [h_1,h_2,\ldots,h_{n-2}]$ are strictly nonlinear functions of x,y and $\bar{z}$. The analysis begins by expanding the components z_i of $\bar{z}$ in power series in x and y, as in eq.(2). This and the other computations may be accomplished more efficiently and accurately by using computer algebra [4,5]. Collecting terms and solving for coefficients as in the example gives an approximate expression for the center manifold, which, when substituted into (14.1) and (14.2), yields an approximation for the flow on the center manifold. Next a perturbation method is used to obtain an approximate solution for $x = x_{lc}$ and $y = y_{lc}$ on the limit cycle in the center manifold. Substitution of these into the expressions derived for the center manifold gives $\bar{z} = \bar{z}_{lc}$ on the limit cycle. The stability of the limit cycle and the center manifold may be obtained by setting:

$$\bar{z} = \bar{z}_{lc} + \delta\bar{z} \tag{15}$$

Substitution of (15) into (14.3) and linearizing in $\delta\bar{z}$ gives the linear variational equations:

$$\delta\dot{\bar{z}} = [A(\mu) + D\bar{h}(x_{lc},y_{lc},\bar{z}_{lc},\mu)]\ \delta\bar{z} \tag{16}$$

in which $D\bar{h}$ is the Jacobian matrix of partial derivatives of h_i with respect to z_j. The elements of $D\bar{h}$ are in general periodic functions of time, and so (16) represents a problem in Floquet theory [6]. The latter may be solved by numerical integration over a single period of the limit cycle, or may be treated by the usual asymptotic methods, cf. Mathieu's equation [5,6].

Summary

The application of center manifold techniques involves a reduction in the dimension of a dynamical system by restricting attention to an invariant subspace (the center manifold) which contains all of the essential behavior. This is the usual procedure for dealing with Hopf bifurcations, where the dimension of a system may be reduced from n to 2. In this work we have extended this procedure by noting that the n-2 dimensions that are usually discarded contain significant information concerning the attractivity, i.e., the stability, of the center manifold itself, as well as of the limit cycle which it contains. The resulting instability of the limit cycle has been seen to give rise to period-doubling.

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References

1. J.E.Marsden and M.McCracken
   The Hopf Bifurcation and Its Applications
   Applied Mathematical Sciences, 19
   Springer-Verlag, New York (1976)

2. J.Guckenheimer and P.Holmes
   Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields
   Applied Mathematical Sciences, 42
   Springer-Verlag, New York (1983)

3. J.Carr
   Applications of Centre Manifold Theory
   Applied Mathematical Sciences, 35
   Springer-Verlag, New York (1981)

4. R.H.Rand and D.Armbruster
   Perturbation Methods, Bifurcation Theory and Computer Algebra
   Applied Mathematical Sciences, 65
   Springer-Verlag, New York (1987)

5. R.H.Rand
   Computer Algebra in Applied Mathematics: An Introduction to MACSYMA
   Research Notes in Mathematics, 94
   Pitman Publishing, Boston (1984)

6. J.J.Stoker
   Nonlinear Vibrations
   Wiley Interscience, New York (1950)