

Chapter 6

A Perturbation Approach to Resonant Capture

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Abstract—This chapter involves resonant capture in systems with a small parameter ε . When $\varepsilon = 0$, the unperturbed system is assumed to contain a separatrix loop; that is, a region bounded by saddle connections. For $\varepsilon > 0$, the saddle connections are broken and motions can become “captured” in the interior of the separated region.

We offer an approximate method for estimating which initial conditions lead to capture, based on invariant manifold theory and perturbation techniques. This method is applied to a model problem consisting of a pendulum under slowly varying torque.

6.1 INTRODUCTION

The capture of trajectories into resonance plays an important role in physical systems arising in many fields of study. In space sciences, it has been used to explain the 3:2 resonance between the spin of the planet Mercury and its orbit around the Sun [Goldreich and Peale 1966], as well as the phase-locking of dual-spin spacecraft [Rand *et al.* 1992], [Hall and Rand 1992]. This phenomenon of resonant capture occurs in dynamical systems containing one or more *resonance manifolds*. For most of the phase space, orbits can be accurately approximated by the averaged equations. However, in the neighborhood of a resonance manifold, a singularity in the averaged equations causes the approximation to fail. A trajectory may pass through the resonance manifold and continue through the rest of phase

space, or it may remain trapped in the neighborhood of the resonance manifold for all time, in which case it is said to be captured.

We restrict our analysis of the dynamics to the neighborhood of the resonance manifold to study this behavior. Typically, the locally valid equations of motion involve a small parameter $\varepsilon \geq 0$. In the unperturbed limit $\varepsilon = 0$, the phase space contains a separatrix, a special trajectory connecting one or more hyperbolic fixed points, and often dividing bounded from unbounded motion. In this limit, orbits cannot cross between these two regions; they either pass through the resonance manifold or remain trapped there for all time. Generally, when $\varepsilon > 0$, no separatrix exists and motions which originate far from the resonance manifold can become trapped there. Moreover, motions which start inside the region occupied by the unperturbed separatrix can, under appropriate conditions, escape from the resonance manifold. In addition, as in the unperturbed system, there generally exist some motions which pass through the resonance manifold.

It is tempting to discuss resonant capture in terms of a curve in phase space corresponding to the “instantaneous separatrix” of the unperturbed system, *i.e.*, the location of the separatrix if ε were suddenly set to zero. Then, under $\varepsilon > 0$ dynamics this “instantaneous separatrix” slowly evolves in time, allowing orbits to cross into the interior of the region and begin to circulate. Trajectories that enter the interior of the separated region are captured, although at some later time they may leave this region, at which point they are said to have escaped capture. It is possible for an orbit to enter and leave the separated region many times, in some cases chaotically. We must emphasize that, while the notion of an “instantaneous separatrix” is a useful way of thinking about capture problems, there is really no such dynamical entity. What actually happens is that, as ε changes from zero to a small nonzero value, the separatrix (a structurally unstable feature) is destroyed (see Figure 6.1).

Thus, the breakup of the separatrix, as shown in Figure 6.1, is responsible for the phenomena of resonant capture. In this perturbed system, the hyperbolic fixed point is replaced by a *normally hyperbolic motion*. Like the hyperbolic fixed point of the unperturbed system, this motion has exponential growth or decay in a neighborhood around it, resulting in *stable and unstable manifolds* which are asymptotic to the trajectory. However, unlike a fixed point, the normally hyperbolic motion generally moves in phase space (although the motion along this orbit is slow). The stable and unstable manifolds of the hyperbolic equilibria, which were once joined together to form the separatrix, split and are no longer identical. Trajectories are allowed to cross into the region of phase space once defined by the separatrix. Thus, the calculation of the stable and unstable manifolds of the normally hyperbolic motion reveals which initial conditions lead to capture and which lead to pass-through.

The separatrix of the unperturbed system is structurally unstable and almost any $\varepsilon > 0$ perturbation destroys it. However, if $0 < \varepsilon \ll 1$, the stable and unstable manifolds of the normally hyperbolic motion are “close” to the separatrix of the unperturbed system. We represent these critical trajectories as a power series in ε and require that the solutions at each order in ε approach the normally

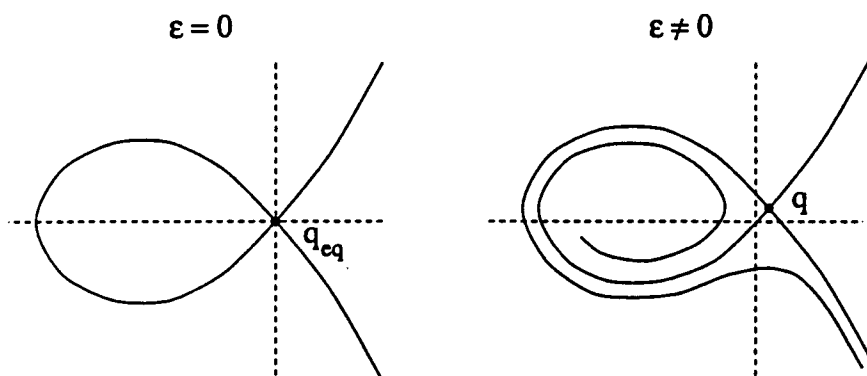


Figure 6.1. Destruction of the separatrix in phase space for $\varepsilon > 0$. The separatrix, which for $\varepsilon = 0$ connects the hyperbolic saddle point q_{eq} to itself, is replaced by the stable and unstable manifolds of the normally hyperbolic motion q . In our system, the stable and unstable manifolds are two-dimensional surfaces in the three-dimensional phase space. These sketches represent their intersection with a plane transverse to the normally hyperbolic motion.

hyperbolic motion as $t \rightarrow \infty$. The lowest-order approximation is the separatrix of the unperturbed problem.

6.1.1 Dynamical System

Consider the general equations:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, \eta) + \varepsilon \mathbf{f}_1(\mathbf{x}, \eta) + \cdots + \varepsilon^i \mathbf{f}_i(\mathbf{x}, \eta) + O(\varepsilon^{i+1}), \quad (6.1)$$

$$\dot{\eta} = \varepsilon r_1(\mathbf{x}, \eta) + \cdots + \varepsilon^i r_i(\mathbf{x}, \eta) + O(\varepsilon^{i+1}). \quad (6.2)$$

$\mathbf{x} \in \mathbb{R}^2$, $\eta \in \mathbb{R}$. When $\varepsilon = 0$, this reduces to:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}, \eta), \quad (6.3)$$

$$\dot{\eta} = 0, \quad (6.4)$$

which will be referred to as the unperturbed system. In addition, we require that Eqs. (6.3) and (6.4) satisfy the following:

Saddle Connection Assumption. Equations (6.3) and (6.4) are assumed to have a hyperbolic saddle point $\mathbf{x}(\eta) = \mathbf{q}_{eq}(\eta)$ with a saddle connection having known solution $\mathbf{x}_0(t, \eta)$.

6.1.2 Model Problem

As an example of resonant behavior, we study the simple pendulum shown in Figure 6.2, subject to a small, slowly varying torque $\varepsilon\eta$, or $\ddot{x} + \sin x = \varepsilon\eta$. If we assume that the applied torque exponentially decays to a constant value $\varepsilon\eta_\infty$, the equations of motion for this system can be written as:

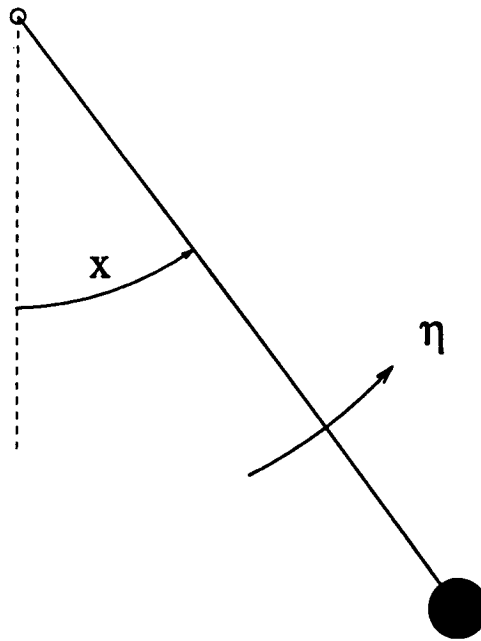


Figure 6.2. Model System: A simple one-degree-of-freedom pendulum subject to a torque η which exponentially decays to a constant value η_∞ as $t \rightarrow \infty$.

$$\dot{x} = y, \quad (6.5)$$

$$\dot{y} = -\sin x + \varepsilon\eta, \quad (6.6)$$

$$\dot{\eta} = \varepsilon(\eta_\infty - \eta). \quad (6.7)$$

Note that Eq. (6.7) can be solved exactly to yield:

$$\eta(t) = (\eta^* - \eta_\infty) \exp(-\varepsilon t) + \eta_\infty,$$

with the initial condition:

$$\eta|_{t=0} = \eta^*.$$

The corresponding unperturbed system is simply an unforced pendulum, given by:

$$\dot{x} = y,$$

$$\dot{y} = -\sin x,$$

$$\dot{\eta} = 0.$$

In this limit, equilibria exist at:

$$x^{(eq)} = \{0, \pm\pi\},$$

$$y^{(eq)} = 0$$

In addition, the unperturbed system has an exact solution in terms of elliptic functions. However, for this analysis, we only need the solution along the unperturbed separatrix, which consists of a pair of orbits, homoclinic to the fixed point at $x = \pm\pi$, shown in Figure 6.3 and described by:

$$x_{\pm}^{sep} = \pm 2 \arctan(\sinh \tau),$$

$$y_{\pm}^{sep} = \pm 2 \operatorname{sech} \tau.$$

$$\tau = t - t_0$$

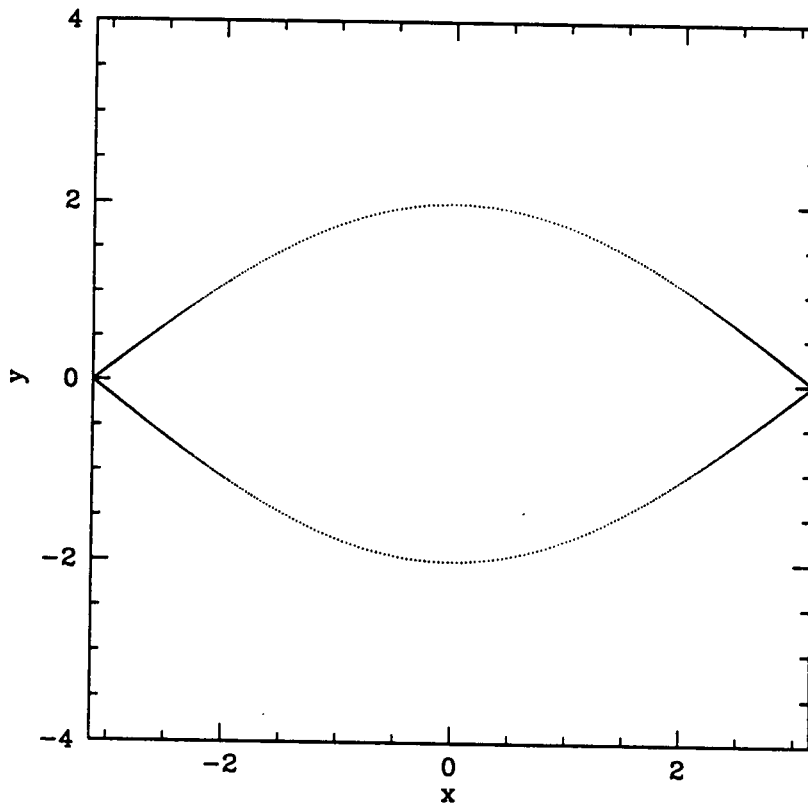


Figure 6.3. Unperturbed separatrix of an undamped, unforced pendulum.

For this problem, the resonance manifold is $y = 0$. Capture occurs in a region of phase space around $y = 0$, in which the separatrix of the unperturbed system is located. For motions with large angular velocities $|\dot{x}| \gg 1$, *i.e.*, those which start far from the separatrix in the cylindrical phase space, the effect of gravity is negligible, and the $\sin x$ term in Eq. (6.6) may be neglected. However, as a motion approaches the resonance manifold, the presence of the separatrix becomes important, and capture is possible.

Capture for this problem may be described in words as follows: we imagine a motion which is initially librating in a direction opposite to the applied torque. As

time goes by, the torque reduces the angular velocity and eventually reverses the direction of the motion. If the applied torque were constant, this motion would then begin to librate in the direction of the torque. However, because the applied torque is not constant, this trajectory might begin to oscillate, corresponding to a captured orbit. Alternatively, as in the case of the constant-torque system, it could librate in the direction of the applied torque and pass through the resonance manifold.

To observe the resonant capture phenomena in Eqs. (6.5) to (6.7), we have numerically integrated the model system and marked those initial conditions that remain bounded, as shown in Figure 6.4. In contrast to these numerical results, in the following section we develop analytical approximations for those initial conditions which lead to capture.

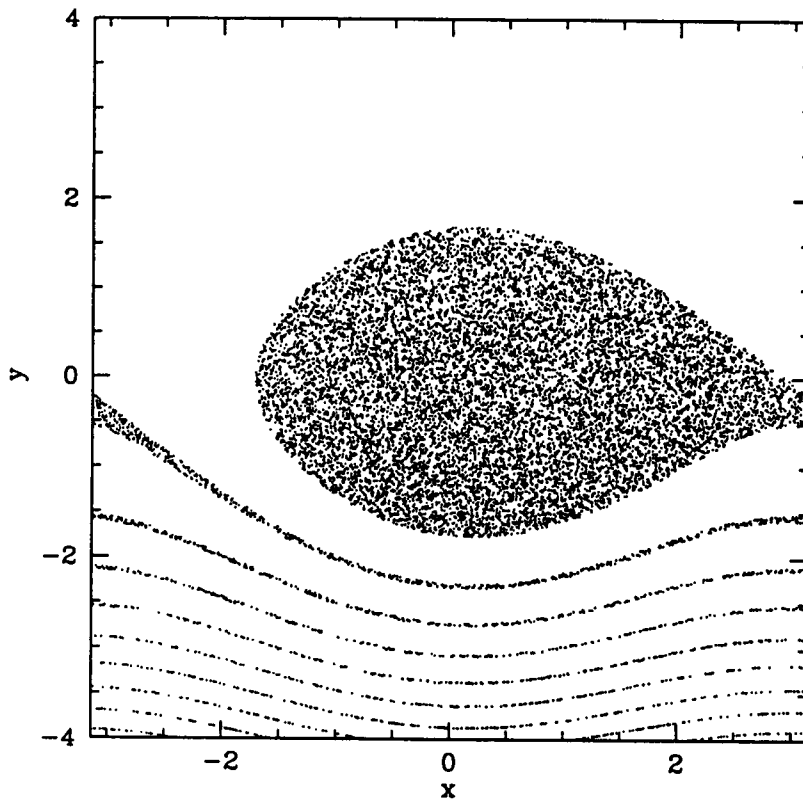


Figure 6.4. Numerically generated region of capture for $\eta_\infty = 1$, $\epsilon = 0.1$, and the initial condition $\eta|_{t=0} = 2$. Each marked point corresponds to an initial condition which leads to capture; all other points escape. In this figure a trajectory is considered captured once it begins to oscillate rather than rotate. Thus, captured trajectories are bounded in forward time.

6.2 INVARIANT MANIFOLD CONSTRUCTION

We seek to calculate the critical trajectories which divide captured and escaped motions. These trajectories are simply the stable and unstable manifolds of the normally hyperbolic motion of Eqs. (6.1) and (6.2). The asymptotic construction

of these manifolds is based on results found in [Guckenheimer and Holmes 1983], [Robinson 1983] which can be summarized as follows:

1. If Eqs. (6.1) and (6.2) satisfies the Saddle Connection Assumption, then there exists a normally hyperbolic motion M_ε , denoted by $q(t)$, for $0 < \varepsilon \ll 1$. In the limit $\varepsilon = 0$, $q(t) \rightarrow q_{\varepsilon q}(\eta)$, the saddle point. Motion along M_ε is slow, of $O(\varepsilon)$. Moreover, M_ε possesses a stable (unstable) manifold, $W^{s(u)}(\varepsilon)$, of points which approach M_ε exponentially fast as t goes to $\infty(-\infty)$.
2. Orbits $\mathbf{x}^{s(u)}(t; t_0)$ lying on $W^{s(u)}(\varepsilon)$ can be expressed as follows (valid on the indicated time interval):

$$\begin{aligned} \mathbf{x}_\varepsilon^s(t; t_0) &= \mathbf{x}_0(t - t_0) + \varepsilon \mathbf{x}_1^s(t; t_0) + O(\varepsilon^2), & t \in [t_0, \infty) \\ \mathbf{x}_\varepsilon^u(t; t_0) &= \mathbf{x}_0(t - t_0) + \varepsilon \mathbf{x}_1^u(t; t_0) + O(\varepsilon^2), & t \in (-\infty, t_0] \end{aligned}$$

where $\mathbf{x}_0(t - t_0)$ is the solution to the unperturbed system on the separatrix. Note that $\mathbf{x}_0(t - t_0)$ is invariant to time translations because Eqs. (6.3) and (6.4) is autonomous.

Thus, the existence of these critical orbits is guaranteed, as well as their structure. In the remainder of this section, we present a method for calculating $W^{s(u)}(\varepsilon)$ of Eqs. (6.1) and (6.2). These manifolds are two-dimensional surfaces in the three-dimensional $\{\mathbf{x}, \eta\}$ phase space and can be parameterized by $\{t_0, \eta^*\}$, the two initial conditions associated with the motion. In addition to the time-history of the manifolds, we can obtain the intersection of these manifolds with planes of constant η by setting $t = 0$ and then varying t_0 in $\mathbf{x}_\varepsilon^s(0; t_0)$, the expression for orbits lying on the stable manifold of M_ε . Thus, we obtain the intersection of the manifolds with the plane $\eta = \eta^*$. The resulting intersections will be a curve in \mathbf{x} -space which separates those initial conditions that are captured from those that escape through the resonance zone. This is the analytical analogue of Figure 6.4.

Although we have restricted our attention to \mathbf{R}^2 , the extension of this calculation to \mathbf{R}^n is not difficult, provided one can solve a system of n first-order nonhomogeneous linear differential equations. For the sake of brevity, we will present the derivation only to $O(\varepsilon)$. However, the higher-order calculations at $O(\varepsilon^i)$ with $i \geq 2$ are straightforward.

6.2.1 Normally Hyperbolic Motion

The normally hyperbolic motion M_ε of Eqs. (6.1) and (6.2) is analogous to the hyperbolic saddle point of Eqs. (6.3) and (6.4). Geometrically, the expansion and contraction rates normal to M_ε dominate over those rates tangent to this trajectory. The existence of a normally hyperbolic motion in Eqs. (6.1) and (6.2) is guaranteed if the Saddle Connection Assumption holds. We seek to construct this orbit via a perturbation approach, together with invariant manifold theory.

The general solution of Eqs. (6.1) and (6.2) can be written as:

$$\mathbf{x} = \mathbf{x}(\eta(t), t),$$

so that $\dot{\mathbf{x}}$ is given as:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \mathbf{x}}{\partial \eta} \frac{d\eta}{dt} \tag{6.8}$$

In particular, we are interested in the normally hyperbolic motion, which we denote by the choice of variables $\{\mathbf{x}, \eta\} \rightarrow \{\mathbf{q}, \mu\}$.

The normally hyperbolic motion is an invariant manifold, *i.e.*, a graph over μ [Proposition 3.3] [Robinson 1983]; thus, it has no explicit time dependence and we write $\mathbf{q}(\mu(t))$. By restricting our interest to the normally hyperbolic motion, Eq. (6.1) reduces to:

$$\frac{d\mathbf{q}}{dt} = \frac{d\mathbf{q}}{d\mu} \frac{d\mu}{dt},$$

or, using Eq. (6.2):

$$\frac{d\mathbf{q}}{dt} = \frac{d\mathbf{q}}{d\mu} \{ \epsilon r_1(\mathbf{q}, \mu) + O(\epsilon^2) \}. \tag{6.9}$$

To calculate $\mathbf{q}(\mu)$, we transform the independent variable of Eq. (6.1) from $t \rightarrow \mu$ using (6.9), resulting in:

$$\{ \epsilon r_1(\mathbf{q}, \mu) \} \frac{d\mathbf{q}}{d\mu} = \mathbf{f}_0(\mathbf{q}, \mu) + \epsilon \mathbf{f}_1(\mathbf{q}, \mu) + O(\epsilon^2), \tag{6.10}$$

a two-dimensional system, nonautonomous in μ , governing the normally hyperbolic motion $\mathbf{q}(\mu)$.

Because ϵ multiplies the highest-order derivative of $\mathbf{q}(\mu)$, a general solution would require singular perturbation techniques. However, the Saddle Connection Assumption assures us that $\mathbf{q}_{\epsilon q}(\mu)$ is nondegenerate and regular perturbation methods can be applied.

Expand $\mathbf{q}(\mu)$ in ϵ :

$$\mathbf{q}(\mu) = \mathbf{q}_0(\mu) + \epsilon \mathbf{q}_1(\mu) + \epsilon^2 \mathbf{q}_2(\mu) + \dots$$

and substitute into Eq. (6.10). Because of the singular nature of Eq. (6.10), derivatives of $\mathbf{q}(\mu)$ are introduced at higher order in ϵ and the calculation of $\mathbf{q}(\mu)$ reduces to solving algebraic equations.

Collecting terms in ϵ :

$$\begin{aligned} O(\epsilon^0) : \mathbf{f}_0(\mathbf{q}_0, \mu) &= 0, \\ O(\epsilon^1) : \mathbf{f}_1(\mathbf{q}_0, \mu) &= r_1(\mathbf{q}_0, \mu) \frac{d\mathbf{q}_0}{d\mu} - \mathbf{q}_1 \frac{\partial \mathbf{f}_0(\mathbf{q}_0, \mu)}{\partial \mu}. \end{aligned}$$

Comparison with Eq. (6.3) shows that $\mathbf{q}_0(\mu)$ corresponds to the hyperbolic fixed point of the unperturbed system:

$$\mathbf{q}_0(\mu) = \mathbf{q}_{eq}(\mu)$$

and $\mathbf{q}(\mu)$ is $O(\varepsilon)$ close to this saddle point. This follows from the explicit independence of t in $\mathbf{q}(\mu(t))$. Once $\mathbf{q}_0(\mu)$ is known, $\mathbf{q}_1(\mu)$ is again found by solving an algebraic equation at $O(\varepsilon^1)$.

To determine $\mu(t)$, return $\mathbf{q}(\mu(t))$ to Eq. (6.2), resulting in:

$$\frac{d\mu}{dt} = \varepsilon r_1(\mathbf{q}_0 + \varepsilon \mathbf{q}_1 + O(\varepsilon^2), \mu) + O(\varepsilon^2). \quad (6.11)$$

and again apply regular perturbation methods to determine the time history of μ . Expand $\mu(t)$ in ε as before:

$$\mu(t) = \mu_0(t) + \varepsilon \mu_1(t) + \varepsilon^2 \mu_2(t) + \dots,$$

substitute into Eq. (6.11) and collect terms in ε :

$$\begin{aligned} O(\varepsilon^0) : \dot{\mu}_0 &= 0, \\ O(\varepsilon^1) : \dot{\mu}_1 &= r_1(\mathbf{q}_0(\mu_0), \mu_0), \end{aligned}$$

subject to the initial condition:

$$\mu|_{t=0} = \mu^*. \quad (6.12)$$

We solve at each order in ε for $\mu(t)$, but for the moment leave μ^* unspecified. This initial condition will be determined below.

To calculate the stable and unstable manifolds of the normally hyperbolic motion, we require \mathbf{q} as an explicit function of t . Thus, we substitute $\mu(t)$ into $\mathbf{q}(\mu)$ and reexpand in ε to obtain:

$$\begin{aligned} O(\varepsilon^0) : \hat{\mathbf{q}}_0(t) &= \mathbf{q}_0(\mu_0(t)) \\ O(\varepsilon^1) : \hat{\mathbf{q}}_1(t) &= \frac{d\mathbf{q}_0}{d\mu}(\mu_0(t))\mu_1(t) + \mathbf{q}_1(\mu_0(t)). \end{aligned}$$

so that the normally hyperbolic motion \mathbf{q} can be written as:

$$\hat{\mathbf{q}}(t) = \hat{\mathbf{q}}_0(t) + \varepsilon \hat{\mathbf{q}}_1(t) + \dots$$

where the hats represent \mathbf{q} as an explicit function of t , in contrast to \mathbf{q} as a function of μ . However, for simplicity, we will drop the hats in the remaining calculations and view \mathbf{q} only as a function of t .

Although the normally hyperbolic motion $\mathbf{q}(t)$ is analogous to the hyperbolic fixed point of the unperturbed system, $\dot{\mathbf{q}}(t)$ is no longer identically equal to zero. This orbit is not an equilibrium point, but an invariant manifold which remains in the neighborhood of $\mathbf{q}_{eq}(\mu)$.

6.2.2 Stable and Unstable Manifolds of the Normally Hyperbolic Motion

We again use regular perturbation techniques to construct the stable and unstable manifolds, $W^s(\varepsilon)$ and $W^u(\varepsilon)$, of M_ε , the normally hyperbolic motion.

As $t \rightarrow \infty$ ($-\infty$), the stable (unstable) manifold $W^{s(u)}(\varepsilon)$ approaches M_ε . However, M_ε is generally not stationary; points on M_ε flow in time. Therefore, the manifolds do not approach a point in phase space; they approach an orbit. We shift the origin of Eqs. (6.1) and (6.2) to follow the normally hyperbolic motion through the transformation:

$$\mathbf{x}(t) = \mathbf{q}(t) + \mathbf{u}(t) \quad \eta(t) = \mu(t) + \omega(t) \quad (6.13)$$

where $\{\mathbf{q}(t), \mu(t)\}$ are again the coordinates of M_ε . Thus $\mathbf{u}(t)$ and $\omega(t)$ describe the motion in phase space relative to the normally hyperbolic motion. Because $\{\mathbf{q}(t), \mu(t)\}$ satisfy Eqs. (6.1) and (6.2), substitution of Eq. (6.13) into Eqs. (6.1) and (6.2) results in:

$$\begin{aligned} \dot{\mathbf{u}} &= \mathbf{f}_0(\mathbf{q} + \mathbf{u}, \mu + \omega) - \mathbf{f}_0(\mathbf{q}, \mu) \\ &\quad + \varepsilon \{\mathbf{f}_1(\mathbf{q} + \mathbf{u}, \mu + \omega) - \mathbf{f}_1(\mathbf{q}, \mu)\} + O(\varepsilon^2), \\ \dot{\omega} &= \varepsilon \{r_1(\mathbf{q} + \mathbf{u}, \mu + \omega) - r_1(\mathbf{q}, \mu)\} + O(\varepsilon^2). \end{aligned} \quad (6.14)$$

Expressing \mathbf{u} and ω in a series in ε ,

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}_0(t) + \varepsilon \mathbf{u}_1(t) + \varepsilon^2 \mathbf{u}_2(t) + \dots, \\ \omega(t) &= \omega_0(t) + \varepsilon \omega_1(t) + \varepsilon^2 \omega_2(t) + \dots, \end{aligned}$$

we substitute these expansions into Eq. (6.14) and replace $\{\mathbf{q}(t), \mu(t)\}$ by their expansions as determined from the analysis of the normally hyperbolic motion. Collecting terms in ε yields:

$$O(\varepsilon^0) : \dot{\mathbf{u}}_0(t) = \mathbf{f}_0(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0) - \mathbf{f}_0(\mathbf{q}_0, \mu_0), \quad (6.15)$$

$$\dot{\omega}_0(t) = 0, \quad (6.16)$$

$$O(\varepsilon^1) : \dot{\mathbf{u}}_1(t) = \mathbf{f}_1(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0) - \mathbf{f}_1(\mathbf{q}_0, \mu_0) \quad (6.17)$$

$$\begin{aligned} &+ \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0)(\mathbf{u}_1 + \mathbf{q}_1) - \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}(\mathbf{q}_0, \mu_0)(\mathbf{q}_1) \\ &+ \frac{\partial \mathbf{f}_0}{\partial \eta}(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0)(\omega_1 + \mu_1) - \frac{\partial \mathbf{f}_0}{\partial \eta}(\mathbf{q}_0, \mu_0)(\mu_1), \end{aligned}$$

$$\dot{\omega}_1(t) = r_1(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0) - r_1(\mathbf{q}_0, \mu_0). \quad (6.18)$$

Asymptotic Behavior of $W^{s(u)}(\varepsilon)$. If $\{\mathbf{x}(t), \eta(t)\} \in W^{s(u)}(\varepsilon)$, then as $t \rightarrow \infty$ ($-\infty$), $\{\mathbf{x}(t), \eta(t)\} \rightarrow \{\mathbf{q}(t), \mu(t)\}$.

The transformation Eq. (6.13) and the asymptotic behavior of $W^{s(u)}(\varepsilon)$ require that $\{\mathbf{u}(t), \omega(t)\} \rightarrow \{0, 0\}$ as $t \rightarrow \infty$ ($-\infty$). As a result:

$$\omega_0 = 0.$$

Because q_0 and μ_0 are constants and satisfy equilibrium equations, the $O(\varepsilon^0)$ solution $\{u_0, \omega_0\}$ corresponds to the motion around the separatrix of the unperturbed system. The Saddle Connection Assumption guarantees that this motion is known. Because Eqs. (6.2) and (6.3) is autonomous, u_0 takes the form:

$$u_0(t; t_0) = u_0(t - t_0).$$

where t_0 is an arbitrary time shift.

With $\{u_0, \omega_0\}$ known, Eq. (6.18) reduces to:

$$\dot{\omega}_1(t) = F_1(t)$$

where $F_1(t)$ is a known function. Thus, $\omega_1(t)$ can be found by quadrature. Again, the asymptotic behavior of $W^{*(u)}(\varepsilon)$ requires that $\omega_1(t) \rightarrow 0$ as $t \rightarrow \infty$ ($-\infty$). However, this equation exhibits only one constant. If we specify the asymptotic behavior of $\omega_1(t)$, then we are no longer able to choose the initial conditions. However, because we declined to fix the initial condition of $\mu(t)$ earlier in the derivation, we can now require:

$$\begin{aligned} \mu|_{t=0} &= \eta|_{t=0} - \omega|_{t=0}, & \text{or} \\ \mu^* &= \eta^* - \omega|_{t=0}. \end{aligned} \quad (6.19)$$

That is, we select the initial condition on $\mu(t)$ to satisfy:

1. An arbitrary initial condition on $\eta(t)$,
2. The necessary initial conditions on $\omega(t)$ such that $\omega(t)$ has the desired asymptotic behavior.

With the lowest-order solution known, Eq. (6.17), the first variational equation, can be written as:

$$\dot{u}_1(t) = L(t)u_1 + N_1(t),$$

where:

$$L(t) = \frac{\partial f_0}{\partial x}(q_0 + u_0, \mu_0 + \omega_0),$$

and $N_1(t)$ is the nonhomogeneous component of the system. $N_1(t)$ depends on not only t but $u_0(t)$ as well. However, $u_0(t)$ is a known vector. The general solution to this equation takes the form:

$$u_1(t) = c_{11}u_{11}(t) + c_{12}u_{12}(t) + u_{1p}(t),$$

where $u_{11}(t)$ and $u_{12}(t)$ are linearly independent complementary solutions and $u_{1p}(t)$ is the corresponding particular solution. We note that Vakakis has presented a strategy for finding the general solution $u_1(t)$, starting by first finding the

complementary solution $\mathbf{u}_{11}(t)$ from $\mathbf{u}_0(t)$ [Vakakis 1994], [Rand 1994]. Observe that, because $\{\mathbf{q}_0, \mu_0\}$ correspond to the hyperbolic saddle point:

$$\frac{d}{dt} \{\dot{\mathbf{u}}_0 = \mathbf{f}_0(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0)\}$$

reduces to:

$$\begin{aligned} \ddot{\mathbf{u}}_0 &= \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0) \dot{\mathbf{u}}_0, \\ &= \mathbf{L}(t) \dot{\mathbf{u}}_0, \end{aligned}$$

where $\mathbf{L}(t)$ is defined above. Therefore, by inspection, we obtain one complementary solution:

$$\mathbf{u}_{11} = \dot{\mathbf{u}}_0.$$

With one solution known, \mathbf{u}_{12} can be found by reduction of order. Once both complementary solutions are known, \mathbf{u}_{1p} may be found by variation of parameters. The arbitrary constants c_{11} and c_{12} are chosen to ensure that $\mathbf{u}_1(t) \rightarrow 0$ as $t \rightarrow \infty$ ($-\infty$) for $W^s(u)(\varepsilon)$. The resulting expressions on \mathbf{u} and ω represent the stable and unstable manifolds of M_ε .

6.2.3 Higher-Order Calculations

The higher-order calculations of both the normally hyperbolic motion and its stable and unstable manifolds are analogous to those at order ε . However, we mention some features of the method that simplify the calculations. At $O(\varepsilon^i)$ the variational equation can be written as:

$$\begin{aligned} \dot{\mathbf{u}}_i(t) &= \mathbf{L}_i(t) \mathbf{u}_i + \mathbf{N}_i(t), \\ \dot{\omega}_i(t) &= F_i(t). \end{aligned}$$

where $F_i(t)$ and $\mathbf{N}_i(t)$ depend on the lower-order solutions of $(\mathbf{u}_k(t), \omega_k(t))$, which are assumed known. $\mathbf{N}_i(t)$ also depends on $\omega_i(t)$, but this can always be found by quadrature. We also note that $\mathbf{L}_i(t)$ is identical at each order in ε :

$$\mathbf{L}_i(t) = \frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}(\mathbf{q}_0 + \mathbf{u}_0, \mu_0 + \omega_0).$$

Therefore, at $O(\varepsilon^i)$ the complementary solution $\mathbf{u}_{C_i}(t)$ reduces to:

$$\mathbf{u}_{C_i}(t) = c_{i1} \mathbf{u}_{11}(t) + c_{i2} \mathbf{u}_{12}(t),$$

where \mathbf{u}_{11} and \mathbf{u}_{12} are determined in the $O(\varepsilon)$ analysis. In Appendix A, we present a general solution for $W^s(\varepsilon)$, when the unperturbed system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ g(x) \end{bmatrix}.$$

6.3 APPLICATION TO A PENDULUM WITH SLOWLY VARYING TORQUE

We now apply the method of invariant manifolds to our model problem of section 6.1.2. As an example of the higher-order calculations, we include terms of $O(\varepsilon^2)$. Comparison of our model Eqs. (6.5) to (6.7) with Eqs. (6.1) and (6.2) gives:

$$\mathbf{f}_0 = \begin{bmatrix} y \\ -\sin x \end{bmatrix},$$

$$\mathbf{f}_1 = \begin{bmatrix} 0 \\ \eta \end{bmatrix},$$

$$r_1 = \eta_\infty - \eta.$$

The invariant manifold of the normally hyperbolic motion $\{q, p, \mu\}$ satisfies the equations:

$$\varepsilon(\eta_\infty - \mu) \frac{dq}{d\mu} = p$$

$$\varepsilon(\eta_\infty - \mu) \frac{dp}{d\mu} = -\sin q + \varepsilon\mu$$

corresponding to Eq. (6.10) in the derivation.

Expanding $\{q, p\}$ in ε results in the following system of algebraic equations governing the normally hyperbolic motion:

$$\begin{aligned} O(\varepsilon^0) : 0 &= p_0 \\ 0 &= -\sin q_0 \end{aligned}$$

$$\begin{aligned} O(\varepsilon^1) : (\eta_\infty - \mu) \frac{dq_0}{d\mu} &= p_1 \\ (\eta_\infty - \mu) \frac{dp_0}{d\mu} &= \mu - q_1 \cos q_0 \end{aligned}$$

$$\begin{aligned} O(\varepsilon^2) : (\eta_\infty - \mu) \frac{dq_1}{d\mu} &= p_2 \\ (\eta_\infty - \mu) \frac{dp_1}{d\mu} &= -2q_2 \cos q_0 \end{aligned}$$

The equilibrium point at $q = \pi$ is of saddle-type and is, therefore, taken as the zero-order solution. This system can now be solved at each order in ε to yield:

$$\begin{aligned} q(\mu) &= \pi - \varepsilon\mu + O(\varepsilon^3) \\ p(\mu) &= \varepsilon^2(\mu - \eta_\infty) + O(\varepsilon^3) \end{aligned}$$

As noted in the introductory discussion of the model system, the time evolution of μ is governed by the one-dimensional equation:

$$\frac{d\mu}{dt} = \varepsilon (\eta_\infty - \mu),$$

which can be solved in closed form to yield:

$$\mu(t) = (\mu^* - \eta_\infty) \exp(-\varepsilon t) + \eta_\infty, \quad (6.20)$$

where:

$$\mu^* = \mu(0).$$

Although a closed-form solution for $\mu(t)$ is convenient, it is not necessary for the application of this method. Following our derivation, we expand μ in ε :

$$\frac{d\mu_0}{dt} + \varepsilon \frac{d\mu_1}{dt} + \varepsilon^2 \frac{d\mu_2}{dt} = \varepsilon (\eta_\infty - \mu_0) - \varepsilon^2 \mu_1 + O(\varepsilon^3).$$

Solving for $\mu_i(t)$ at each order in ε results in:

$$\mu(t) = c + (\eta_\infty - c)\varepsilon t - (\eta_\infty - c) \frac{(\varepsilon t)^2}{2} + O(\varepsilon^3).$$

With the proper choice of the arbitrary constant c and the initial condition, this reduces to:

$$\mu(t) = \eta_\infty + (\mu^* - \eta_\infty) \left(1 - \varepsilon t + \frac{(\varepsilon t)^2}{2} \right) + O(\varepsilon^3), \quad (6.21)$$

We note that to $O(\varepsilon^3)$, Eq. (6.21) is identical to the Taylor series expansion of Eq. (6.20). Following our derivation, for the moment we leave as unspecified the initial condition of Eq. (6.21):

$$\mu|_{t=0} = \mu^*,$$

(cf. Eqs. (6.12) and (6.19)).

Substituting Eq. (6.21) into $(q(\mu), p(\mu))$, and reexpanding in ε , the time history of the normally hyperbolic motion takes the form:

$$q(t) = \pi - \varepsilon \mu^* + \varepsilon^2 (\mu^* - \eta_\infty) t + O(\varepsilon^3), \quad (6.22)$$

$$p(t) = \varepsilon^2 (\mu^* - \eta_\infty) + O(\varepsilon^3). \quad (6.23)$$

Next, we look for the stable manifold of this motion through the transformation:

$$x = q + u,$$

$$y = p + v,$$

$$\eta = \mu + \omega$$

where $\{q, p, \mu\}$ are known from above. Through this transformation, the origin of this new system (in the new variables $\{u, v, \omega\}$) coincides with the normally hyperbolic motion and Eqs. (6.5)–(6.7) reduce to:

$$\begin{aligned}\frac{du}{dt} &= v \\ \frac{dv}{dt} &= -\sin(q + u) + \sin q + \varepsilon\omega \\ \frac{d\omega}{dt} &= -\varepsilon\omega\end{aligned}$$

Expanding $\{u, v, \omega\}$ in ε and replacing $\{q, p, \mu\}$ by Eqs. (6.22) and (6.23), we obtain, to each order in ε :

$$\begin{aligned}O(\varepsilon^0): \frac{du_0}{dt} &= v_0 \\ \frac{dv_0}{dt} &= \sin u_0 \\ \frac{d\omega_0}{dt} &= 0\end{aligned}$$

$$\begin{aligned}O(\varepsilon^1): \frac{du_1}{dt} &= v_1 \\ \frac{dv_1}{dt} &= u_1 \cos u_0 + \omega_0 + \mu^*(1 - \cos u_0) \\ \frac{d\omega_1}{dt} &= -\omega_0\end{aligned}$$

$$\begin{aligned}O(\varepsilon^2): \frac{du_2}{dt} &= v_2 \\ \frac{dv_2}{dt} &= u_2 \cos u_0 + \omega_1 - (\tau + t_0)(\mu^* - \eta_\infty)(1 - \cos u_0) - \\ &\quad (u_1 - \mu^*)^2 \sin u_0 \\ \frac{d\omega_2}{dt} &= -\omega_1\end{aligned}$$

From the asymptotic behavior of the stable manifold, solutions at each order in ε must decay to zero as $t \rightarrow \infty$. Thus, the ε^0 solution corresponds to the separatrix of the unperturbed system (with the saddle point moved to the origin):

$$\begin{aligned}u_0 &= -2 \arctan(\sinh \tau) + \pi, \\ v_0 &= -2 \operatorname{sech} \tau, \\ \omega_0 &= 0,\end{aligned}$$

$$\tau = t - t_0.$$

For simplicity, we transform our independent variable from $t \rightarrow \tau$. With this solution, we may now solve $\frac{d\omega_1}{d\tau} = -\omega_0$, subject to the condition $\omega_1 \rightarrow 0$ as $t \rightarrow \infty$, to obtain:

$$\omega_1 = 0,$$

We similarly find that:

$$\omega_2 = 0.$$

As in our derivation, we define $N_i(t)$ to be the nonhomogeneous component of the system at $O(\varepsilon^i)$. Thus, at each order in ε we are left with the linear, nonhomogeneous, time-varying system:

$$\begin{bmatrix} \frac{du_i}{d\tau} \\ \frac{dv_i}{d\tau} \end{bmatrix} = \begin{bmatrix} v_i \\ u_i \cos u_0 + N_i(\tau) \end{bmatrix} \quad \text{for} \quad i = 1, 2 \quad (6.24)$$

where, after some manipulation:

$$\begin{aligned} N_1(\tau) &= 2\mu^* \operatorname{sech}^2 \tau \\ N_2(\tau) &= 2(\tau + t_0)(\eta_\infty - \mu^*) \operatorname{sech}^2 \tau - (u_1 - \mu^*)^2 \sinh \tau \operatorname{sech}^2 \tau \end{aligned} \quad (6.25)$$

As noted previously, the general solution to Eq. (6.24) takes the form:

$$u_i(\tau) = c_{i1}u_{11}(\tau) + c_{i2}u_{12}(\tau) + u_{ip}(\tau)$$

in which we may take:

$$u_{11}(\tau) = \operatorname{sech} \tau,$$

the derivative of the unperturbed solution. To find u_{12} and u_{ip} we use reduction of order, variation of parameters, and the computer algebra system MACSYMA. Expressions for u_{12} and u_{ip} are found in Appendix B. Because the integrals of Eq. (6.28) cannot be completely evaluated in closed form, we rely on numerical quadrature to obtain our results.

Thus, after identifying the initial condition $\mu^* = \eta^*$, we finally obtain expressions for $x(\tau)$ and $\eta(\tau)$ on the stable manifold of the normally hyperbolic motion:

$$x(\tau) = -2 \arctan(\sinh \tau) + \varepsilon \{u_1(\tau) - \eta^*\} + \varepsilon^2 \{u_2(\tau) + (\eta^* - \eta_\infty)(\tau + t_0)\} + O(\varepsilon^3), \quad (6.26)$$

$$\eta(\tau) = \eta_\infty + (\eta^* - \eta_\infty) \left(1 - \varepsilon(\tau + t_0) + \frac{(\varepsilon(\tau + t_0))^2}{2} \right) + O(\varepsilon^3), \quad (6.27)$$

$$\tau = t - t_0$$

where u_1 and u_2 are given in Appendix B. An expression for $y(\tau) = \frac{dx}{d\tau}$ may be obtained by differentiating Eq. (6.26). To compare our results with those of Figure 6.4 we find calculate $x(-t_0)$ as t_0 varies. This corresponds to the intersection of $W^s(\varepsilon)$ with the plane $\eta(t_0) = \eta^*$ which we present in Figure 6.5 for comparison with the numerically obtained results of Figure 6.4.

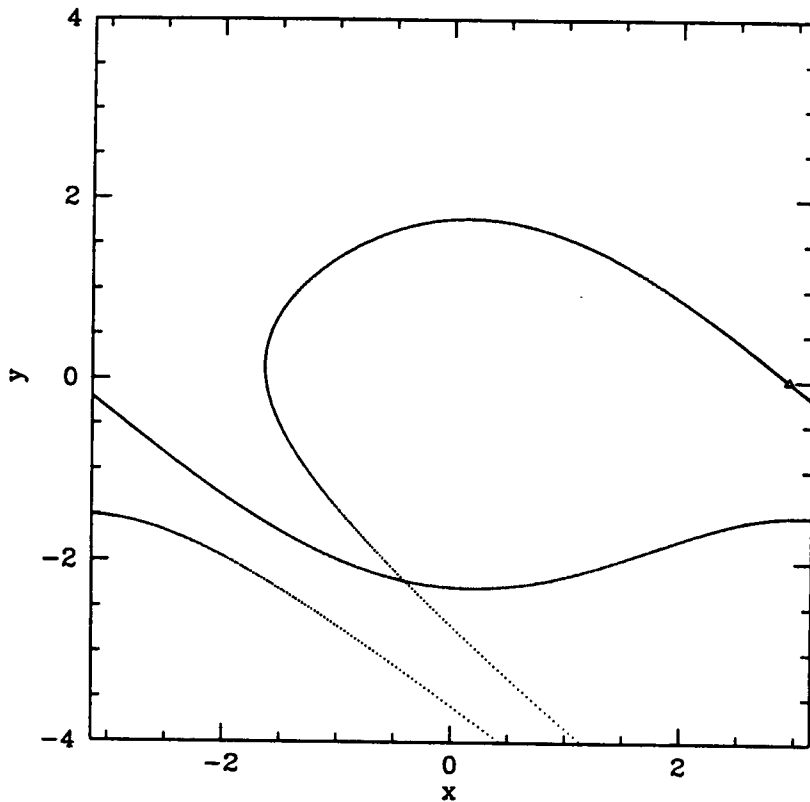


Figure 6.5. Analytical approximation for the intersection of the stable manifold with the plane $\eta = 2$ for parameter values $\eta_{\infty} = 1.0, \epsilon = 0.1$. The intersection of the normally hyperbolic motion with this plane is denoted by Δ . Note the divergence of the analytical approximation for large times.

6.4 CONCLUSION

We have presented a perturbation approach to problems involving resonant capture, characterized by the presence of a separatrix near the resonance manifold in the unperturbed system. For small values of the parameter ϵ , the separatrix is broken, and capture is possible.

To characterize which initial conditions are captured, the critical trajectories dividing capture and pass-through were calculated using perturbation methods. The saddle equilibrium of the unperturbed system is replaced by a hyperbolic motion in the perturbed system, written as a power series in the small parameter ϵ . Although we worked to $O(\epsilon^2)$ as an example of higher-order calculations, the perturbation series can be easily extended to $O(\epsilon^n)$.

Once the hyperbolic motion has been found, a second perturbation expansion was used to find its stable and unstable manifolds. These are surfaces in the perturbed system which approach the normally hyperbolic motion as $t \rightarrow \infty(-\infty)$ and correspond to the separatrix loop in the unperturbed system. In particular, the

stable manifold separates those motions which are eventually captured from those that pass through resonance.

These methods were applied to a model system, consisting of an undamped pendulum subject to an exponentially decaying torque. Consider a motion which at $t = 0$ is librating in a direction opposite to the applied torque. Due to the effects of the torque, it slows down and eventually reverses its direction. If $\varepsilon = 0$, then the applied torque is constant, and trajectories which initially librate in a direction opposite to the applied torque eventually end up librating in a direction with the torque. However, if $\varepsilon > 0$ a trajectory that initially librates may, because of the non-constant torque, begin to oscillate (and become captured).

It is interesting to note that the phase portrait of a pendulum with constant torque,

$$\ddot{x} + \sin x = \varepsilon\eta, \quad \eta = \text{constant}, \quad \varepsilon\eta < 1,$$

admits a separatrix, see Figure 6.6. Thus orbits cannot cross this special trajectory and capture is not possible. Therefore, it is the slowly changing nature of the applied torque which permits capture in the problem.

An extension of this work could include perturbing off a system with $O(1)$ torque, for example:

$$\begin{aligned} \ddot{x} + \sin x &= \eta, \\ \dot{\eta} &= \varepsilon(\eta_\infty - \eta), \end{aligned}$$

in which case the unperturbed phase portrait would look like Figure 6.6. In this case however, no analytic representation of the unperturbed motion around the separatrix is available, and so the details of such a treatment would involve a hybrid computation which is both numerical and analytical.

6.5 APPENDIX A: $O(\varepsilon^i)$ SOLUTION

We present the general solution for the stable and unstable manifolds of the normally hyperbolic motion of Eqs. (6.1) and (6.2) when the $O(\varepsilon^0)$ component of the vector field can be written as:

$$f_0 = \begin{bmatrix} y \\ g(x) \end{bmatrix}$$

With this form of the unperturbed system, each variational equation on the stable and unstable manifold reduces to:

$$\begin{aligned} \frac{du_i}{d\tau} &= v_i, \\ \frac{dv_i}{d\tau} &= L(\tau)u_i + N_i(\tau), \end{aligned}$$

where:

$$L(\tau) = \frac{dg}{dx}(q_0 + x_0, \mu_0 + \omega_0),$$

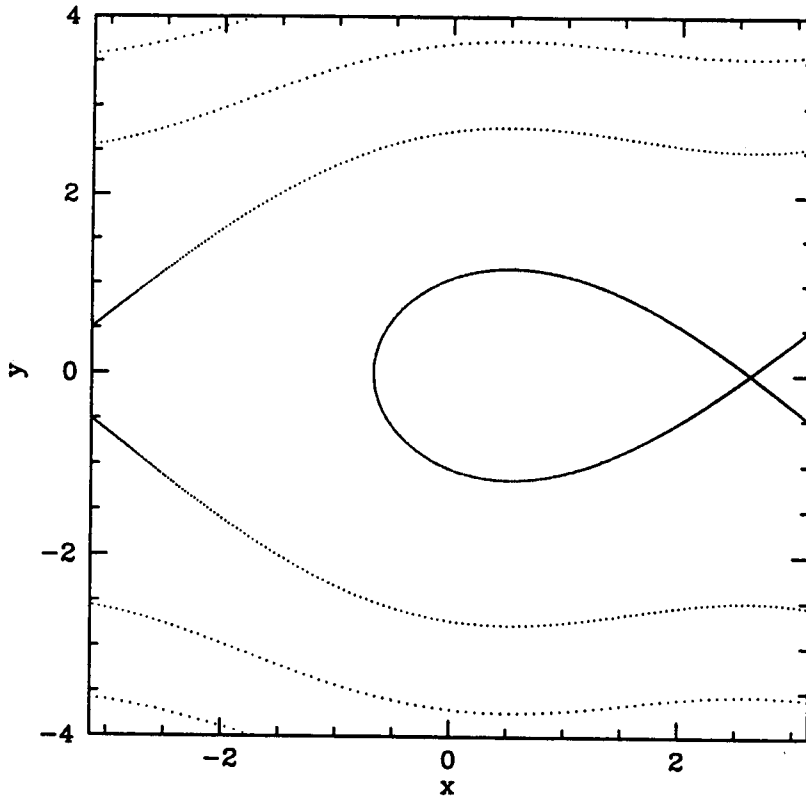


Figure 6.6. The stable and unstable manifolds of the hyperbolic saddle point in the undamped pendulum subject to constant torque, $\ddot{x} + \sin x = 0.5$. The unperturbed separatrix defines the tear-shaped region.

and where $N_i(\tau)$ is the nonhomogeneous component of the system at $O(\varepsilon^i)$ and depends on the lower-order solutions. Because the homogeneous equation at each order in ε is identical, the complementary solution u_C , takes the form:

$$u_{C_i} = c_{i1}u_{11} + c_{i2}u_{12},$$

where, from our derivation we find [Vakakis 1994], [Rand 1994]:

$$u_{11} = \frac{du_0}{d\tau}.$$

We look for a second linearly independent solution of the form:

$$u_{12}(\tau) = \psi(\tau)u_{11}(\tau)$$

By applying reduction of order, we find the second linearly independent solution takes the form:

$$u_{12}(\tau) = u_{11} \int (u_{11})^{-2} d\tau.$$

Once the complementary solution is known, to obtain a particular solution at $O(\varepsilon^i)$, we assume:

$$u_{ip} = \alpha(\tau)u_{11}(\tau) + \beta(\tau)u_{12}(\tau)$$

and find, using variation of parameters:

$$u_{ip}(\tau) = u_{12}(\tau) \int_{-t_0}^{\tau} u_{11}(s)N_i(s)ds - u_{11}(\tau) \int_{-t_0}^{\tau} u_{12}(s)N_i(s)ds$$

Thus, the general solution at $O(\varepsilon^i)$ is:

$$u_i(\tau) = u_{12}(\tau) \left\{ \int_{-t_0}^{\tau} u_{11}(s)N_i(s)ds + c_{i2} \right\} + u_{11}(\tau) \left\{ c_{i1} - \int_{-t_0}^{\tau} u_{12}(s)N_i(s)ds \right\}$$

To obtain solutions on the stable manifold of M_ε , we require that as $\tau \rightarrow \infty$, $u_i(\tau) \rightarrow 0$. To ensure this behavior, we must choose the arbitrary constants c_{i1} and c_{i2} appropriately. As noted by Vakakis [Vakakis 1994], c_{i1} corresponds to an $O(\varepsilon)$ time shift, which we take to be zero. Thus, we set:

$$\begin{aligned} c_{i1} &= 0, \\ c_{i2} &= - \int_{-t_0}^{\infty} u_{11}(s)N_i(s)ds \end{aligned}$$

By combining these constants with the general solution, we form the solution on the stable manifold at each order in ε :

$$u_i(\tau) = -u_{12}(\tau) \int_{\tau}^{\infty} u_{11}(s)N_i(s)ds - u_{11}(\tau) \int_{-t_0}^{\tau} u_{12}(s)N_i(s)ds$$

So that to $O(\varepsilon^n)$, orbits of $W^s(\varepsilon)$ take the form:

$$u(\tau) = u_0(\tau) - \sum_{i=1}^n \left\{ u_{12}(\tau) \int_{\tau}^{\infty} u_{11}(s)N_i(s)ds + u_{11}(\tau) \int_{-t_0}^{\tau} u_{12}(s)N_i(s)ds \right\} \varepsilon^i$$

6.6 APPENDIX B: CALCULATION OF THE STABLE MANIFOLD

Because f_0 of Eqs. (6.5) to (6.7) takes the form:

$$f_0 = \begin{bmatrix} y \\ -\sin x \end{bmatrix}$$

we can follow the general solution of Appendix A:

The complementary solution u_{11} to Eq. (6.24) is found to be:

$$\begin{aligned} u_{11}(\tau) &= \frac{du_0}{d\tau} \\ &= \operatorname{sech} \tau \end{aligned}$$

So that the second complementary solution u_{12} can be found by the integral:

$$\begin{aligned} u_{12}(\tau) &= u_{11} \int (u_{11})^{-2} d\tau \\ &= \frac{1}{2} \{ \sinh \tau + \tau \operatorname{sech} \tau \} \end{aligned}$$

As in Appendix A, the particular solution of Eq. (6.24) at $O(\epsilon^i)$ is written as:

$$u_{ip}(\tau) = u_{12}(\tau) \int_{-t_0}^{\tau} u_{11}(s) N_i(s) ds - u_{11}(\tau) \int_{-t_0}^{\tau} u_{12}(s) N_i(s) ds$$

Thus, choosing the constants c_{i1} and c_{i2} as in Appendix A, we obtain the solution of the stable manifold at $O(\epsilon^i)$:

$$\begin{aligned} u_i(\tau) &= -\frac{1}{2} \{ \sinh \tau + \tau \operatorname{sech} \tau \} \int_{\tau}^{\infty} \{ \operatorname{sech} \tau N_i(s) \} ds - \\ &\quad \operatorname{sech} \tau \int_{-t_0}^{\tau} \left\{ \frac{1}{2} \{ \sinh \tau + \tau \operatorname{sech} \tau \} N_i(s) \right\} ds \end{aligned} \quad (6.28)$$

where $N_i(\tau)$ is given in Eq. (6.25). These integrals cannot be completely evaluated in closed form. We obtained values for them by using numerical quadrature.

6.7 REFERENCES

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