Numerical Corrections of Wu's Coefficients for Scattering of High-Frequency Waves from Spheres and Cylinders

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The asymptotic expansions derived for high-frequency scattering by cylinders and spheres are reexamined in detail. It is found that earlier results require small but significant corrections. The accuracy of the new analytical results is confirmed by comparison with numerical evaluations.

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The theory of scattering owes much to Wu's pioneering work on the subject. In particular his well-known formulas[1-3] for high-frequency scattering from spheres and cylinders are of fundamental importance and are also widely used. Recently some small discrepancies between those formulas and numerical calculations (R. T. Waechter, private communication) provided us with the incentive to reexamine Wu's theory and the accuracy of his results.

Wu's results[1] are expressed in terms of coefficients M_n and $\overline{M}_n$. Following Wu's notations,[1] M_n, for instance, is defined as

$$M_n = \lim_{R\to\infty}\left\{\frac{(-1)^{n+1}R^{n+1}}{n+1} - \int_{-R}^{\infty}\frac{e^{-i2\pi/3}\,\mathrm{Ai}(v)}{\mathrm{Ai}(e^{i2\pi/3}v)}v^n\,dv\right\}. \tag{1}$$

We take $R=a_j$, where $-a_j$ is the jth root of the Airy function Ai, and close the contour as indicated in Fig. 1. The segment $(-R,R)$ leads to the integral in Eq. (1) as $R\to\infty$. The circle of radius R is composed of Γ_1 for $0<\arg v<\pi/3$, and Γ_3 for $\pi/3<\arg v<\pi$. Γ_1 and Γ_3 are joined by a semicircle Γ_2 with a vanishingly small radius. Asymptotic values for Ai show that the contribution from Γ_1 to the integral is exponentially small as $j\to\infty$ and that

$$\int_{\Gamma_3}\frac{e^{-i2\pi/3}\,\mathrm{Ai}(v)}{\mathrm{Ai}(e^{i2\pi/3}v)}v^n\,dv = \frac{(-1)^n}{n+1}a_j^{n+1} + \frac{e^{i(n+1)\pi/3}}{n+1}a_j^{n+1} + o(a_j^{-N}), \tag{2}$$

where $o(a_j^{-N})$ is smaller in order than any negative power of a_j. Thus

$$M_n = \lim_{j\to\infty}\left\{e^{i(n+1)\pi/3}\frac{a_j^{n+1}}{n+1} - 2\pi i\sum_{k=1}^{j}{}' R_k\right\}, \tag{3}$$

where R_k is the residue at the simple pole $v_k = a_k e^{i\pi/3}$ and the prime on the summation indicates that only half the residue is taken for the last term in the series. Since

$$2\pi i R_k = e^{i(n+1)\pi/3}\alpha_{n,k}, \qquad (4)$$

where

$$\alpha_{n,k} = a_k^n/[\mathrm{Ai}'(a_k)]^2, \qquad (5)$$

we have

$$M_n e^{-i(n+1)\pi/3} = \lim_{j\to\infty}\left\{\frac{a_j^{n+1}}{n+1} - \sum_{k=1}^{j}{}' \alpha_{n,k}\right\}. \qquad (6)$$

For k large, asymptotic results[4] for a_k and $\mathrm{Ai}'(a_k)$ yield

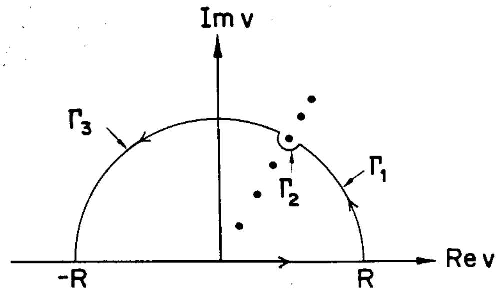

FIG. 1. Sketch of the contour used to estimate the integral in Eq. (1). The dots represent the roots of $\mathrm{Ai}(e^{i2\pi/3}v)$.

$$\alpha_{n,k} = \pi z^{(2n-1)/3}\left\{1 + \left[-\frac{5}{24} + \frac{5}{48}n\right]z^{-2} + \left[\frac{25}{36} - \frac{85}{1152}n + \frac{25}{4608}n(n-1)\right]z^{-4} + \left[\frac{-77\,125}{10\,368} + \frac{85\,525}{82\,944}n - \frac{575}{36\,864}n(n-1) + \frac{125}{663\,552}n(n-1)(n-2)\right]z^{-6} + \dots\right\}, \qquad (7)$$

where

$$z = (3\pi/8)(4k-1). \qquad (8)$$

The Euler-Maclaurin formula may now be applied to Eq. (6) yielding

$$M_n e^{-i(n+1)\pi/3} = \frac{a_r^{n+1}}{n+1} - \sum_{k=1}^{r-1}\alpha_{n,k} - \frac{1}{2}\alpha_{n,r} + \frac{1}{12}\alpha'_{n,r} - \frac{1}{720}\alpha'''_{n,r} + \frac{1}{30\,240}\alpha^{(5)}_{n,r} + \dots, \qquad (9)$$

where the derivatives of $\alpha_{n,r}$ are calculated from Eq. (7). The value of r must be large enough for Eq. (7) to be accurate; in practive $r = 6$ is a suitable value, which we used in our calculations.

We gave the details of the calculation of M_n, as those were not given by Wu.[1] An identical method was followed to obtain $\overline{M}_n$.

Table I gives the values of M_n and $\overline{M}_n$ obtained by Wu and those we calculated here. Our values are given up to the last accurate digit based on the approximations used in the calculations. To that accuracy we note that Wu's values differ significantly from ours. We may also note that $\overline{M}_{-5}$ and $\overline{M}_{-8}$ given by Wu[1] have two terms with different phases; we have corrected them in our Table I.

As an example of the application of Wu's theory let us consider the infiltration from a spherical cavity into a porous medium, a problem which is formally analogous to the scattering problem. The steady flux, Q, can be written as[5]

$$\frac{2Q}{R_0} = \frac{\pi}{R_0^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{I_{n+1/2}(R_0)}{K_{n+1/2}(R_0)}, \qquad (10)$$

where R_0 is proportional to the radius of the cavity and $I_{n+1/2}$ and $K_{n+1/2}$ are the standard modified Bessel functions.[4] With use of elementary properties of the Bessel function the series of Eq. (10) is essentially that in Eq. (5.1) of Wu's paper. Thus in the important limit $R_0 \gg 1$, the right-hand side of Eq. (10) can be rewritten as the real part of

$$1 + 2M_0\left(\frac{R_0}{2}\right)^{-2/3} - \frac{16}{15}M_1\left(\frac{R_0}{2}\right)^{-4/3} - \left(\frac{4}{175}M_2 + \frac{23}{1680}\right)\left(\frac{R_0}{2}\right)^{-2} + 2\left(\frac{64}{70\,875}M_3 + \frac{2}{1575}M_0\right)\left(\frac{R_0}{2}\right)^{-8/3} + 2\left(\frac{2944}{3\,274\,425}M_4 + \frac{334}{363\,825}M_1\right)\left(\frac{R_0}{2}\right)^{-10/3} + O\left[\left(\frac{R_0}{2}\right)^{-4}\right], \qquad (11)$$

which is consistent with Eq. (5.10) of Wu's paper[1] except for the term $\frac{23}{1680}$, which is inadvertently replaced by $\frac{23}{420}$ in Wu's series. With this minor correction values of M_n, either ours or Wu's in Table I, can be used to calculate $2Q/R_0$ for various values of R_0. These are shown in Table II. Results are given to the expected accuracy con-

TABLE I. Values of coefficients M_n and $\overline{M}_n$ calculated by Wu (Ref. 1) and in the present paper. The values of $\overline{M}_{-5}$ and $\overline{M}_{-8}$ given by Wu have been corrected to indicate the proper phase.

	Wu's results	Present calculations
$e^{-i\pi/3}M_0$	1.255 074 37	1.255 124 56
$e^{-i2\pi/3}M_1$	0.532 250 36	0.532 2907
M_2	0.093 5216	0.067 717
$e^{-i\pi/3}M_3$	0.772 793	0.736 77
$e^{-i2\pi/3}M_4$	1.0992	1.0772
$e^{-i\pi/3}\overline{M}_0$	−1.088 874 11	−1.088 888 57
$e^{-i2\pi/3}\overline{M}_1$	−0.934 864 91	−0.934 9266
$\overline{M}_2$	−0.107 0199	−0.142 760
$e^{-i2\pi/3}\overline{M}_3$	−0.757 663	−0.894 27
$e^{-i2\pi/3}\overline{M}_4$	−1.1574	−1.4796
$e^{i\pi/3}\overline{M}_{-2}$	−3.704 093 89	−3.704 093 874
$e^{i2\pi/3}\overline{M}_{-3}$	0.416 821 38	0.416 821 909
$\overline{M}_{-4}$	3.175 796 52	3.195 727 602
$e^{i\pi/3}\overline{M}_{-5}$	0.562 815 61	0.562 815 617
$e^{i2\pi/3}\overline{M}_{-6}$	2.065 757 21	2.065 756 480
$e^{i\pi/3}\overline{M}_{-8}$	−1.582 493 57	−1.582 493 575

sistent with that of the M_n's in Table I. Of course differences between the two estimates are small but significant.

To verify the accuracy of the revised formulation we summed the series in Eq. (10) numerically. The numerical evaluation of the series was performed with the computer algebra system MACSYMA.[6] We used MACSYMA because of its ability to (a) perform exact rational arithmetic and (b) evaluate decimal expansions to an arbitrary number of digits. The Bessel functions of half-integer order were calculated exactly in terms of elementary functions[4] and the sum of the first 170 terms of the series was obtained. Using the MACSYMA function BFLOAT[6] we converted each term to floating point numbers accurate to 100 digits. For a given number of decimal digits of precision, more terms of the expansion are necessary the larger the value of R_0. For instance it is necessary to take 25 terms for $R_0 = 10$ and 164 terms for $R_0 = 100$ if fifteen significant digits are required. Table II also indicates these numerical results, with ten decimal places. We see that our results, rather than Wu's, are correct with the expected accuracy.

TABLE II. Values of $2Q/R_0$ obtained as the real part of Eq. (11) by use of Wu's values and our values for M_n. The exact results were obtained by numerical summation of the series in Eq. (10).

R_0	Wu	Ours	Exact
10	1.461 82	1.461 87	1.461 869 1478
20	1.283 419	1.283 437	1.283 437 5401
50	1.150 6527	1.150 6598	1.150 659 8251
100	1.094 009 47	1.094 013 520	1.094 013 5209

We may note in conclusion the practical value of precise numerical calculations. If they had been available to Wu he would have been able to find and correct his small errors. These calculations confirm the reliability of our results.

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