Free Vibrations of a Thin Elastica by Normal Modes

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Abstract. In this work we investigate the existence, stability and bifurcation of periodic motions in an unforced conservative two degree of freedom system. The system models the nonlinear vibrations of an elastic rod which can undergo both torsional and bending modes. Using a variety of perturbation techniques in conjunction with the computer algebra system MACSYMA, we obtain approximate expressions for a diversity of periodic motions, including nonlinear normal modes, elliptic orbits and non-local modes. The latter motions, which involve both bending and torsional motions in a 2:1 ratio, correspond to behavior previously observed in experiments by Cusumano.

Key words: Nonlinear vibrations, normal modes, perturbation methods, bifurcations.

Introduction

The dynamics of a thin elastica, Figure 1, was the subject of the 1990 Ph.D. thesis of Cusumano [1]. Although the free vibrations of the linear system of Figure 1 involve simply a bending mode and a torsional mode, Cusumano [1] showed that when nonlinear effects are included complicated dynamics result, including chaos. One of the most notable nonlinear effects which is observed was the existence of a combination mode which involved both bending and torsional motions. We shall refer to this motion as a non-local mode. In order to better understand the dynamics of the elastica of Figure 1, Cusumano [1] proposed the simplified two degree of freedom model system of Figure 2 which we shall refer to as system S. Here the rotational motion due to the generalized

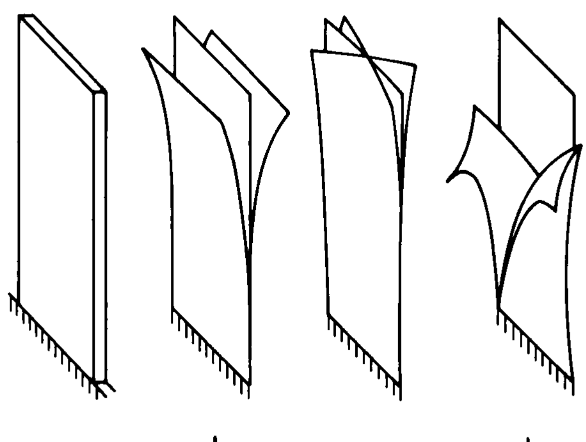

Fig. 1. Thin elastica. (a) undeformed; (b) bending mode; (c) torsional mode; (d) non-local mode, involving both bending and torsion.

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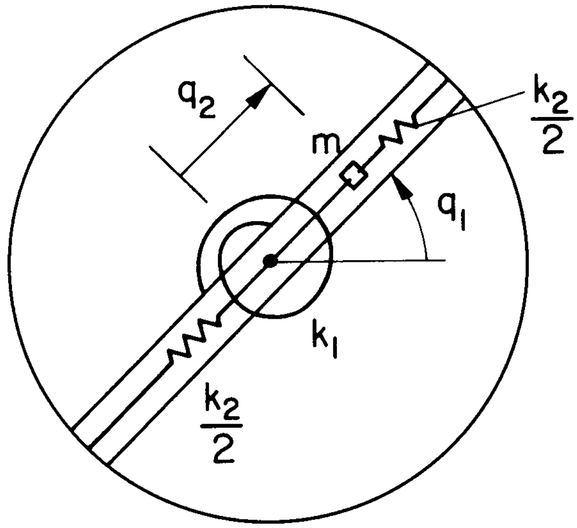

Fig. 2. System S, a simplified two degree of freedom model of the elastica of Figure 1 (Cusumano [1]).

coordinate q_1 is associated with the torsional motion of the elastica, and the rectilinear deflection due to q_2 is associated with the bending motion. When the motion of system S is viewed in the q_1–q_2 configuration plane, the linear normal modes show up as straight-line motions along the coordinate axes, see Figure 3. In addition to the linear normal modes, the non-local mode observed in the elastica is exhibited by system S, appearing as a curved line, Figure 3. Cusumano found that this non-local mode appeared only for sufficiently large energies in the unforced system. In addition he found that the frequency of the non-local mode was inversely proportional to the amplitude of the torsional motion.

In this work we shall extend Cusumano's analysis of system S. In particular we shall show how the non-local mode is related to the linear and nonlinear normal modes of S by using a variety of perturbation methods to investigate the existence, stability and bifurcation of periodic motions of S.

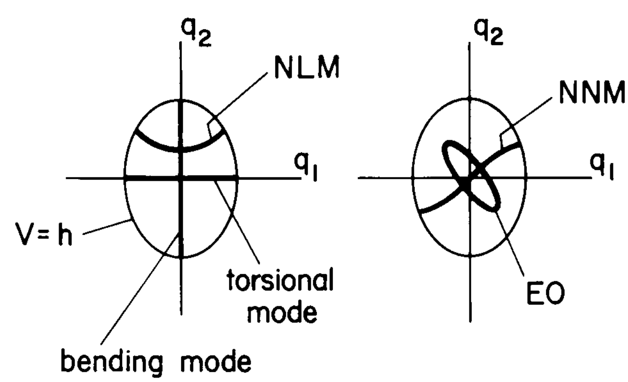

Fig. 3. Various periodic motions of system S viewed in the $q_1 - q_2$ configuration plane: nonlinear normal mode (NNM), elliptic orbit (EO) and non-local mode (NLM). Since $T + V = h =$ constant, and $T \geq 0$, the motion remains in the region $V \leq h$.

Equations of Motion

From Figure 2, the kinetic and potential energies of system S are:

$$T = \frac{m}{2}\,[q_2^2 q_1'^2 + q_2'^2] + \frac{J}{2}\,q_1'^2\,, \tag{1}$$

$$V = \frac{1}{2}\,[k_1 q_1^2 + k_2 q_2^2]\,, \tag{2}$$

where primes represent differentiation with respect to time τ. Then setting

$$x = \sqrt{\frac{Jk_1}{m}}\,q_1\,, \qquad y = \sqrt{k_1}\,q_2\,, \qquad t = \sqrt{\frac{k_1}{m}}\,\tau\,, \tag{3}$$

we obtain the Lagrangian L in dimensionless coordinates:

$$L = \frac{1}{2}\,(1 + \varepsilon y^2)\dot{x}^2 + \frac{1}{2}\,\dot{y}^2 - \frac{1}{2}\,(p^2x^2 + y^2)\,, \tag{4}$$

where

$$\varepsilon = \frac{m}{Jk_1}\,, \qquad p^2 = \frac{k_2/J}{k_1/m} \tag{5}$$

and where dots represent differentiation with respect to nondimensional time t. Note that p is the ratio of the frequencies of the linear normal modes, the torsional x-mode to the bending y-mode.

Lagrange's equations for equation (4) become:

$$(1 + \varepsilon y^2)\ddot{x} + 2\varepsilon y\dot{y}\dot{x} + p^2x = 0\,, \tag{6}$$
$$\ddot{y} - \varepsilon\dot{x}^2y + y = 0\,.$$

Each of the nonlinear terms in the system (6) have as a factor the coefficient ε, and thus arise from the variable inertia term in the Lagrangian (4). Equation (6) may be classified as a two degree of freedom inertially coupled nonlinear system.

We note that the x-mode, $y \equiv 0$, and the y-mode, $x \equiv 0$, are exact solutions. When pictured in the $x - y$ configuration plane, these modes plot as straight lines (and they are the only such straight-line motions, called "similar normal modes" by Rosenberg [10].) The system (6) exhibits symmetry about the x- and y-axes, as well as symmetry about the origin $x = y = 0$ in the configuration plane.

The system (6) is conservative with the energy integral

$$\frac{1}{2}\,(1 + \varepsilon y^2)\dot{x}^2 + \frac{1}{2}\,\dot{y}^2 + \frac{1}{2}\,(p^2x^2 + y^2) = h\,, \tag{7}$$

where h is a constant of the motion.

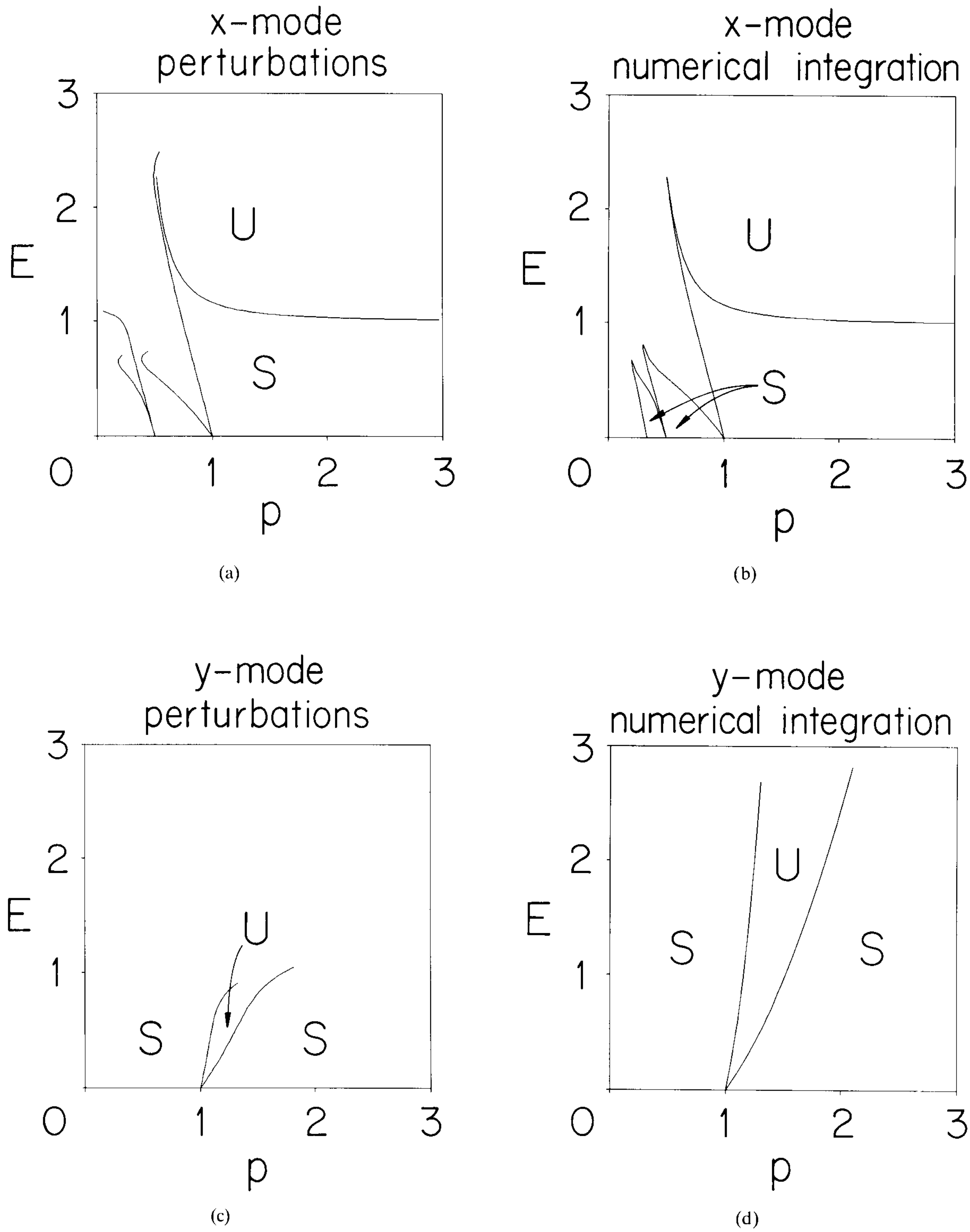

Fig. 4. Stability charts and transition curves for x- and y-modes obtained by perturbations (see Appendix) as well as by numerical integration and Floquet theory. S = Stable, U = Unstable. $E = \varepsilon h$. The x-mode charts do not display the tongues of instability which emerge from $\varepsilon = 0$, $p = \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n}, \cdots$.

Summary of Results

For the reader's convenience, we now present a summary of our results, delaying presentation of the details until later.

We begin by considering the stability of the exact solutions $y \equiv 0$ and $x \equiv 0$. We used perturbations in conjunction with Floquet theory, checking our work by comparison with numerical integration, in order to obtain the results shown in Figure 4 for $0 \leq p \leq 3$ and for $0 \leq \varepsilon h \leq 3$. For $\varepsilon = 0$, tongues of instability for the x-mode $y \equiv 0$ emerge from $p = 1, 1/2, 1/3, \ldots, 1/n, \ldots$ of which only the first two are shown. In similar fashion, for $\varepsilon = 0$, tongues of instability for the y-mode $x \equiv 0$ emerge from $p = 1, 3, 5, \ldots$, although all but the $p = 1$ tongue are very narrow for small ε.

As usual in bifurcation theory, we expect the birth of new periodic motions to accompany the changes in stability of the x- and y-modes. In order to investigate these additional periodic motions, we used a perturbation method for p near unity, valid for small ε. The results are shown in Figure 5. The periodic motions which bifurcate out of the x- and y-modes are of two types. One type, *nonlinear normal modes*, plot as curved line segments in the x-y configuration plane. This type of orbit has been the subject of numerous investigations, especially by Rosenberg and his colleagues [10]. Another type of periodic motion which appears in this problem plots as an ellipse in the x-y plane and hence will be called an *elliptic orbit* [9]. As shown in Figure 5, two stable elliptic orbits and two unstable nonlinear normal modes bifurcate out of the y-mode (the bending mode) for $p > 1$ via a pitchfork bifurcation. In a similar fashion, two stable elliptic orbits and two unstable nonlinear normal modes bifurcate out of the x-mode (the torsional mode) for $p < 1$, also via a pitchfork bifurcation.

Thus each of the stability transition curves in Figure 4 which emanate from $p = 1$ for small εh gives rise to the birth of a pair of periodic motions. Further results may be obtained by looking for periodic solutions associated with the other transition curves in Figure 4. In particular we considered the transition curve for the x-mode which does not touch the p axis (see Figure 4). This curve is the only x-mode transition curve which exists for $p > 1$. It exists only for $\varepsilon h > 1$. We used

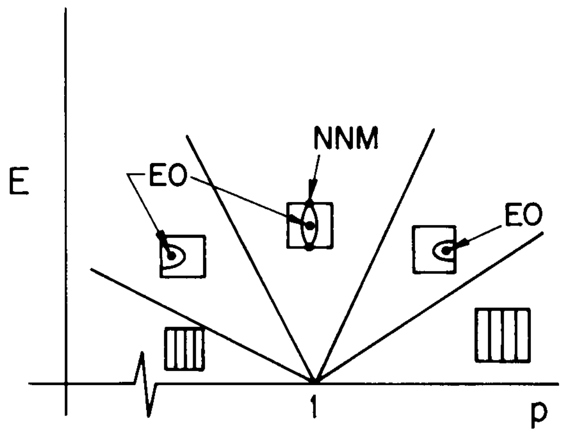

Fig. 5. Schematic slow-flow plots showing periodic motions as singular points for values of p near unity and small $E = \varepsilon h$. NNM = nonlinear normal mode, EO = elliptic orbit. Cf. Figure 6. The transition curves are approximated by $|\Delta|/R^2 = \frac{1}{4}$ and $\frac{3}{4}$, which, with $h = \frac{R^2}{2}$ and $p^2 = 1 + \varepsilon\Delta$, become $p = 1 \pm \frac{1}{4}E$ and $p = 1 + \frac{3}{4}E$. Each of the elliptic orbits has another elliptic orbit (not shown) associated with it, see Figure 6.

Lindstedt's method and the method of harmonic balance to investigate periodic motions near this transition curve, valid for large p. We found a periodic motion which we shall refer to as a *non-local mode*. Two such stable non-local modes bifurcate out of the x-mode via a pitchfork bifurcation as we cross the said transition curve in Figure 4. The natural frequency of the non-local modes are shown to be nearly inversely proportional to the amplitude of the torsional x-mode. These modes correspond to the mode which Cusumano observed in his experiment. They only exist for sufficiently large energies.

We also investigated periodic motions near the transition curves emanating from $p = 1/2$, $\varepsilon h = 0$. Although these modes are qualitatively similar in shape to the non-local mode, these modes exist for small energies and thus are not the ones observed by Cusumano.

Stability of the Torsional and Bending Modes

In order to investigate the stability of the (torsional) x-mode, $y \equiv 0$, we linearize equations (6) about the exact solution:

$$x = A \sin pt + \delta x , \qquad y = 0 + \delta y , \tag{8}$$

giving the variational equations:

$$\ddot{\delta x} + p^2 \, \delta x = 0 , \qquad \ddot{\delta y} + (1 - \varepsilon A^2 p^2 \cos^2 pt) \, \delta y = 0 . \tag{9}$$

The second of the equations (9) is recognized as Mathieu's equation [11]. We set $z = 2pt$ and note from (8) and (7) that $h = A^2p^2/2$, permitting us to write the stability equation in the form:

$$\delta y'' + \left[\frac{1 - E}{4p^2} - \frac{E}{4p^2} \cos z \right] \delta y = 0 , \tag{10}$$

where $E = \varepsilon h$ and where primes represent differentiation with respect to z. Equation (10) is in the standard form of Mathieu's equation,

$$\delta y'' + [\Delta + e \cos z] \delta y = 0 , \tag{11}$$

where

$$\Delta = \frac{1 - E}{4p^2} \quad \text{and} \quad e = -\frac{E}{4p^2} . \tag{12}$$

The structure of the transition curves of equation (11) is well-known. Transition curves emanate from points $e = 0$, $\Delta = N^2/4$, where $N = 0, 1, 2, 3, \ldots$, or in terms of E and p, from points $E = 0$, $p = 1/N$. Asymptotic expansions for the transition curves are available via computer algebra [7, 8]. E.g., the transition curve through the origin $\Delta = e = 0$ has the expansion [11]:

$$\Delta = -\frac{1}{2} e^2 + \frac{7}{32} e^4 + \cdots . \tag{13}$$

If equations (12) are substituted into (13) and E is expanded in a power series of $(1/p)$, we obtain:

$$E = 1 + \frac{1}{8}\frac{1}{p^2} + \frac{1}{32}\frac{1}{p^4} + \cdots . \tag{14}$$

Similarly we found the following transition curves through $E = 0$, $p = 1$, and $E = 0$, $p = 1/2$:

$$p^2 = 1 - \frac{1}{2}E + \frac{1}{32}E^2 + \cdots \tag{15}$$

$$p^2 = 1 - \frac{3}{2}E + \frac{1}{32}E^2 + \cdots \tag{16}$$

$$p^2 = \frac{1}{4} - \frac{E}{4} - \frac{5}{48}E^2 + \cdots \tag{17}$$

$$p^2 = \frac{1}{4} - \frac{E}{4} + \frac{1}{48}E^2 + \cdots \tag{18}$$

Using computer algebra, we obtained expressions for the foregoing transition curves valid to higher order, and have presented these in the Appendix.

In a similar fashion, we may investigate the stability of the (bending) y-mode, $x \equiv 0$, by linearizing equations (6) about the exact solution:

$$x = 0 + \delta x , \qquad y = A \sin t + \delta y , \tag{19}$$

giving the variational equations:

$$(1 + \varepsilon A^2 \sin^2 t)\,\delta\ddot{x} + \varepsilon A^2 \sin 2t\,\delta\dot{x} + p^2\,\delta x = 0 , \qquad \delta\ddot{y} + \delta y = 0 . \tag{20}$$

The first of equations (20) is called Ince's equation ([5], p. 2). We set $z = 2t$ and note from (19) and (7) that $h = A^2/2$, permitting us to write it in the form:

$$[1 + E - E\cos z]\,\delta x'' + E \sin z\,\delta x' + \frac{p^2}{4}\,\delta x = 0 , \tag{21}$$

where $E = \varepsilon h$ and where primes represent differentiation with respect to z.

In order to determine values of p and E which lie on transition curves for equation (21), we use the result from Floquet theory [11] that on such curves there exist periodic solutions with the same period ($=2\pi$), or twice the period ($=4\pi$), as the periodic coefficient. When $E = 0$, equation (21) has solutions of period $4\pi/p$. Equating this to $4\pi/N$, for $N = 0, 1, 2, \ldots$ (since solutions with period $4\pi/N$ also have period 4π, and if N is even, have period 2π; $N = 0$ corresponds to a constant, which may be said to have infinite period), we find that equation (21) has transition curves intersecting the p-axis ($E = 0$) at $p = N$. In order to obtain asymptotic expansions for these curves, we substitute

$$p^2 = N^2 + k_1\varepsilon + k_2\varepsilon^2 + \cdots \tag{22}$$

$$\delta x = \delta x_0 + \delta x_1\varepsilon + \delta x_2\varepsilon^2 + \cdots \tag{23}$$

into equation (21) and collect terms. The function δx_0 is separately taken as cos Nt, then as sin Nt, since each choice is expected to give a distinct transition curve. In fact we find that the two such transition curves corresponding to N even coincide, and hence that the associated region of instability does not exist, a situation called "coexistence" [5]. Moreover, the case $N = 0$ corresponds to the exact transition curve $p \equiv 0$ (since the "periodic" solution $\delta x = 1$ exists on this line.) Thus we shall restrict our attention to $N = 1, 3, 5, \ldots$. Removal of secular terms in the δx_i equation (for $i = 1, 2, 3, \ldots$) provides a condition for the unknown coefficient k_i. In this way we find the following expressions for the transition curves through $E = 0$, $p = 1$ and $E = 0$, $p = 3$:

$$p^2 = 1 + \frac{1}{2} E - \frac{9}{32} E^2 + \cdots \tag{24}$$

$$p^2 = 1 + \frac{3}{2} E - \frac{9}{32} E^2 + \cdots \tag{25}$$

$$p^2 = 9\left[1 + E - \frac{23}{64} E^2 + \frac{185}{512} E^3 + \cdots\right] \tag{26}$$

$$p^2 = 9\left[1 + E - \frac{23}{64} E^2 + \frac{183}{512} E^3 + \cdots\right]. \tag{27}$$

Note that the transition curves through $E = 0$, $p = 3$ are very close to each other and hence that the associated region of instability is very small for small E.

Higher order expressions for the transition curves (24), (25), obtained by using computer algebra, are presented in the Appendix.

The transition curve perturbation results presented in this section were checked by numerically integrating equations (10) or (20) and computing the associated characteristic exponents of Floquet theory [11]. The results of this process are compared with the foregoing perturbation results in Figure 4.

Periodic Motions for p Near Unity

The foregoing stability analysis has shown that both the x- and y-modes exhibit changes in stability for small ε when p is near unity. In order to better understand the nature of the dynamics involved in these stability changes, we use the two variable expansion perturbation method [3, 6, 8]. We begin by defining the detuning Δ as:

$$p^2 = 1 + \varepsilon\Delta . \tag{28}$$

Next we replace t as independent variable by the two variables

$$\xi = t \quad \text{and} \quad \eta = \varepsilon t , \tag{29}$$

in which case

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial \xi} + \varepsilon \frac{\partial x}{\partial \eta} , \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\partial^2 x}{\partial \xi^2} + 2\varepsilon \frac{\partial^2 x}{\partial \xi\, \partial \eta} + \varepsilon^2 \frac{\partial^2 x}{\partial \eta^2} \tag{30}$$

with similar expressions for $\dot{y}$ and $\ddot{y}$. Next we expand x and y in power series in ε,

$$x(\xi, \eta, \varepsilon) = x_0(\xi, \eta) + \varepsilon x_1(\xi, \eta) + \cdots$$

and a similar expression for y. Substitution into equations (6) gives, after collecting terms and equating the coefficients of like powers of ε to zero:

$$\frac{\partial^2 x_0}{\partial \xi^2} + x_0 = 0 , \qquad \frac{\partial^2 y_0}{\partial \xi^2} + y_0 = 0 \tag{31}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = -2 \frac{\partial^2 x_0}{\partial \xi\, \partial \eta} - 2y_0 \frac{\partial y_0}{\partial \xi} \frac{\partial x_0}{\partial \xi} - y_0^2 \frac{\partial^2 x_0}{\partial \xi^2} - \Delta x_0 \tag{32}$$

$$\frac{\partial^2 y_1}{\partial \xi^2} + y_1 = -2 \frac{\partial^2 y_0}{\partial \xi\, \partial \eta} + y_0 \left(\frac{\partial x_0}{\partial \xi}\right)^2 \tag{33}$$

We take the solution of equations (31) in the form:

$$y_0 = A(\eta) \cos \xi + \beta(\eta) \sin \xi , \tag{34}$$

$$x_0 = C(\eta) \cos \xi + D(\eta) \sin \xi , \tag{35}$$

where $A(\eta)$, $B(\eta)$, $C(\eta)$, $D(\eta)$ are arbitrary functions of η. We substitute equations (34), (35) into (32), (33), trigonometrically simplify and collect terms, and equate to zero the coefficients of the resonant terms $\cos \xi$ and $\sin \xi$, giving

$$\frac{dA}{d\eta} = -\frac{BD^2}{8} + \frac{ACD}{4} - \frac{3BC^2}{8} \tag{36}$$

and three similar equations on $B(\eta)$, $C(\eta)$, $D(\eta)$. Next we transform to polar coordinates, $R_1(\eta)$, $R_2(\eta)$, $\theta_1(\eta)$, $\theta_2(\eta)$

$$A = R_1 \cos \theta_1 , \quad B = R_1 \sin \theta_1 , \quad C = R_2 \cos \theta_2 , \quad D = R_2 \sin \theta_2 , \tag{37}$$

noting that the zeroth order solution, equations (34), (35), can be written

$$x_0 = R_1(\eta) \cos(\xi - \theta_1(\eta)) , \qquad y_0 = R_2(\eta) \cos(\xi - \theta_2(\eta)) . \tag{38}$$

Substituting equations (37) into the d.e.'s on A, B, C, D (e.g. equation (36)) we obtain d.e.'s on R_1, R_2, θ_1, θ_2 which are naturally written in terms of the variable φ:

$$\varphi = \theta_2 - \theta_1 , \tag{39}$$

$$\frac{dR_1}{d\eta} = \frac{1}{8} R_1 R_2^2 \sin 2\varphi , \tag{40}$$

$$\frac{dR_2}{d\eta} = -\frac{1}{8} R_1^2 R_2 \sin 2\varphi , \tag{41}$$

$$\frac{d\varphi}{d\eta} = -\frac{\Delta}{2} + \frac{1}{4}(R_1^2 - R_2^2)\left(1 - \frac{1}{2}\cos 2\varphi\right). \tag{42}$$

Finally we transform to polars R, ψ in the $R_1 - R_2$ plane,

$$R_1 = R\cos\psi\,, \qquad R_2 = R\sin\psi\,, \tag{43}$$

which replaces equations (40), (41), (42) by the following:

$$\frac{dR}{d\eta} = 0\,, \tag{44}$$

$$\frac{d\psi}{d\eta} = -\frac{1}{16}R^2\sin 2\varphi\sin 2\psi\,, \tag{45}$$

$$\frac{d\varphi}{d\eta} = -\frac{\Delta}{2} + \frac{R^2}{4}\cos 2\psi\left[1 - \frac{1}{2}\cos 2\varphi\right]. \tag{46}$$

From equation (44), we see that R is a constant, a result equivalent to conservation of energy. From equations (26), (27) we obtain the following first integral:

$$\frac{4\Delta}{R^2}\cos 2\psi - \cos^2 2\psi - \frac{1}{2}\cos 2\varphi\sin^2 2\psi = \text{constant}\,. \tag{47}$$

For a given value of Δ/R^2, we may plot the integral curves of equation (47) on the $\psi - \varphi$ phase plane, for $0 < \psi < \pi/2$, $0 \leq \varphi \leq \pi$, see Figure 6. (Although φ is naturally defined on $0 \leq \varphi \leq 2\pi$ from (39), the resulting plot is symmetric about $\varphi = \pi$, and thus the region $\pi \leq \varphi \leq 2\pi$ may be omitted.) Singular points of (47) correspond to equilibria of (45), (46) and to periodic

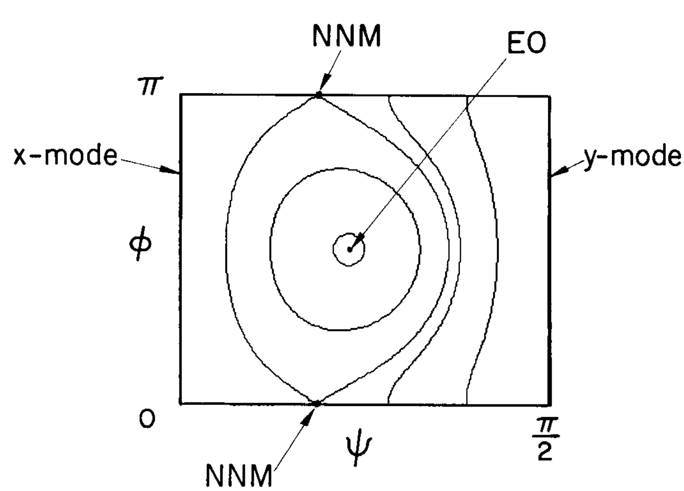

Fig. 6. Plot of approximate first integral (47) for $\Delta/R^2 = \frac{1}{10}$. The y-mode, $x = 0$, corresponds to $R_1 = 0$ from equation (38), and to $\psi = \frac{\pi}{2}$ from equation (43). Similarly the x-mode, $y = 0$, corresponds to $\psi = 0$. Although φ is naturally defined on $0 \leq \varphi \leq 2\pi$, cf. equation (39), the plot is symmetric about $\varphi = \pi$, and thus the region $\pi \leq \varphi < 2\pi$ has been omitted. In particular, the elliptic orbit (EO) shown has another elliptic orbit associated with it, located symmetrically with respect to $\varphi = \pi$, which corresponds to motion around the same orbit in the $x - y$ plane, but in the opposite direction.

motions of (6). Equilibria of (45) require either that $\sin 2\varphi$ or $\sin 2\psi$ vanish. From (43), $\sin 2\psi = 0$ corresponds to the x- and y-modes. On the other hand, $\sin 2\varphi = 0$ corresponds to some other periodic motions. In this case, either φ must equal 0 or π (in which case $x(t)$ and $y(t)$ are in phase or are 180° out of phase, and the periodic motion plots as a line segment in the $x - y$ plane, a nonlinear normal mode) or φ must equal $\pi/2$ or $3\pi/2$ (in which case $x(t)$ and $y(t)$ are 90° out of phase, and the periodic motion plots as an ellipse in the $x - y$ plane, an elliptic orbit).

Turning to equation (46), an equilibrium which corresponds to a nonlinear normal mode satisfies $\cos 2\varphi = 1$, which requires that $\cos 2\psi = 4\Delta/R^2$, i.e., $|\Delta|/R^2 \leq 1/4$. On the other hand, an equilibrium which corresponds to an elliptic orbit satisfies $\cos 2\varphi = -1$, which requires that $\cos 2\psi = 4\Delta/3R^2$, i.e. $|\Delta|/R^2 \leq 3/4$.

The stability of each of these singularities may be determined by linearizing (45), (46) and computing the eigenvalues of the local flow. It turns out that the bifurcating nonlinear normal modes are unstable while the elliptic orbits are stable. The results are shown in Figure 5. Note that for $\Delta > 0$, two unstable nonlinear normal modes bifurcate out of the y-mode at $\Delta/R^2 = 1/4$, and two stable elliptic orbits bifurcate out of the y-mode at $\Delta/R^2 = 3/4$. For $\Delta < 0$, the same events occur with respect to the x-mode. These results complement our stability analysis of the previous section, and show that in the neighborhood of $p = 1$, $\varepsilon = 0$, as a transition curve is crossed (by varying p or ε in a quasistatic fashion) a pair of periodic motions (either nonlinear normal modes or elliptic orbits) is born or dies via a pitchfork bifurcation.

The Non-Local Mode

Motivated by the observation that periodic motions are born as one crosses the transition curves emanating from $p = 1$, $\varepsilon = 0$, we now look for periodic motions in the neighborhood of the x-mode transition curves equation (14), which approaches $E = 1$ in the limit as $p \to \infty$ We shall use Lindstedt's method [8, 11] to accomplish this. We begin by stretching time with the change of independent variable:

$$z = \omega p t\,, \qquad \text{where } \omega = 1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots\,. \tag{48}$$

Then we expand x, y and p^{-2} in power series in ε:

$$x = x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \cdots\,, \qquad y = y_0 + y_1 \varepsilon + y_2 \varepsilon^2 + \cdots\,, \tag{49}$$

$$\frac{1}{p^2} = \mu = \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \cdots\,. \tag{50}$$

Here the constants ω_i and μ_i are to be chosen in order to remove secular terms. Substituting (48)–(50) into equations (6) and collecting terms gives:

$$x_0'' + x_0 = 0\,, \qquad y_0'' = 0\,, \tag{51}$$

$$x_1'' + x_1 = -(2\omega_1 + y_0^2)x_0'' - 2y_0 y_0' x_0'\,, \tag{52}$$

$$y_1'' = [(x_0')^2 - \mu_1] y_0 - 2\omega_1 y_0''\,, \tag{53}$$

as well as similar equations on x_2 and y_2 which we omit for brevity. The periodic solution to the unperturbed equations (51) will be taken as:

$$x_0 = a_1 \cos z + a_2 \sin z\,, \qquad y_0 = a_3\,. \tag{54}$$

Substitution of (54) into (52), (53) leads to the removal of secular terms by the following choice of parameters:

$$\omega_1 = -\frac{a_3^2}{2}\,, \qquad \mu_1 = \frac{a_1^2 + a_2^2}{2}\,. \tag{55}$$

Substitution of (54) and (55) into (52), (53) yields the particular solution:

$$x_1 = 0\,, \qquad y_1 = \frac{a_3}{8}\,[2a_1 a_2 \sin 2z + (a_1^2 - a_2^2)\cos 2z]\,. \tag{56}$$

Substitution of (54)–(56) into the x_2, y_2 equations and removing secular terms gives:

$$\omega_2 = \frac{a_3^2(a_1^2 + a_2^2)}{16} + \frac{3}{8}\,a_3^4\,, \qquad \mu_2 = -\frac{a_3^2(a_1^2 + a_2^2)}{2} - \frac{(a_1^2 + a_2^2)^2}{32}\,. \tag{57}$$

Equations (54) and (56), when substituted into (49), give an asymptotic approximation for a family of periodic motions. If we select a particular phase by choosing the arbitrary constant $a_1 = 0$, we obtain:

$$x = a_2 \sin \omega pt + \cdots\,, \qquad y = a_3\left[1 - \frac{\varepsilon}{8}\,a_2^2 \cos 2\omega pt\right] + \cdots\,, \tag{58}$$

where ω and p are obtained by substituting (55) and (57) into (48) and (50).

This mode, when plotted in the $x - y$ plane for particular choices of a_2 and a_3, looks like the non-local mode which Cusumano observed in his experiments. Note that for given values of p and ε, equations (50), (55) yield a value for a_2 (with $a_1 = 0$). Thus the non-local mode has a single parameter, a_3, associated with it. For given energy h there are a pair of such non-local modes, each the reflection of the other in the x-axis. From (58), the frequency of this non-local mode is given by the approximate expression:

$$\text{frequency} = \omega p = (1 + \omega_1\varepsilon + \cdots)(\mu_1\varepsilon + \cdots)^{-1/2} = \sqrt{\frac{2}{\varepsilon}}\,\frac{1}{a_2} + \cdots\,. \tag{59}$$

that is, the frequency of the non-local mode is nearly inversely proportional to the amplitude a_2 of the torsional mode, in agreement with the findings of Cusumano [1].

Cusumano also found that the non-local mode only occurred for sufficiently large energies. In order to check the behavior of our approximate solution (58) with this observation, we need to compute the energy of (58) by substituting it into the expression (7) for energy h. This gives:

$$h = \frac{1}{\varepsilon} + \frac{1}{16}\,a_2^2 + \frac{3}{2}\,a_3^2 + O(\varepsilon)\,. \tag{60}$$

Note that when $a_3 = 0$, the derived periodic motion (58) reduces to the torsional x-mode. Thus the non-local mode bifurcates out of the x-mode as the energy h increases through the critical value

$$h = \frac{1}{\varepsilon} + \frac{1}{16} a_2^2 + O(\varepsilon) . \tag{61}$$

Equation (61) becomes, when expressed in terms of the parameter $E = \varepsilon h$ used in equation (14) for the stability analysis of the x-mode,

$$E = 1 + \frac{1}{16} \varepsilon a_2^2 + O(\varepsilon^2) . \tag{62}$$

This may be rewritten using (50) and (55), i.e., $p^2 = 2/(a_2^2 \varepsilon) + \cdots$,

$$E = 1 + \frac{1}{8} \frac{1}{p^2} + \cdots , \tag{63}$$

which agrees with the stability equation (14). Thus we have shown that the non-local mode bifurcates out of the x-mode as the latter changes its stability as one crosses the transition curve (14). Since a_3 may be either positive or negative, two such nodes are born together in a pitchfork bifurcation.

Periodic Motions for p Near 1/2

We note that the non-local mode (58) involves a y-motion which occurs at twice the frequency as the accompanying x-motion. It is interesting to compare this mode with the periodic motions which bifurcate out of the x-mode as one crosses the transition curves (17), (18) associated with the point $E = 0$, $p = 1/2$, see Figure 4. Since p is the ratio of the frequencies of the x- to y-modes, we expect to again find periodic motions for which the y-motion occurs at twice the frequency as the x-motion. As we will show, however, these motions do not correspond to the non-local mode observed by Cusumano, because they do not require a minimum energy in order that they occur, and because their frequency is not inversely proportional to the amplitude of the torsional mode.

We again use Linstedt's method, this time setting

$$z = \omega t , \qquad \text{where } \omega = 1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots . \tag{64}$$

Then we expand x, y and p^2 in power series in ε:

$$x = x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \cdots , \qquad y = y_0 + y_1 \varepsilon + y_2 \varepsilon^2 + \cdots , \tag{65}$$

$$p^2 = \frac{1}{4} + c_1 \varepsilon + c_2 \varepsilon^2 + \cdots . \tag{66}$$

Here the constants ω_i and c_i are to be chosen in order to remove secular terms. Substituting (64)–(66) into equations (6) and collecting terms gives:

$$x_0'' + \frac{1}{4} x_0 = 0 , \qquad y_0'' + y_0 = 0 , \tag{67}$$

$$x_1'' + \frac{1}{4}x_1 = -(2\omega_1 + y_0^2)x_0'' - c_1x_0 - 2y_0x_0'y_0', \tag{68}$$

$$y_1'' + y_1 = y_0x_0'^2 - 2\omega_1 y_0'', \tag{69}$$

as well as similar equations on x_2 and y_2 which we omit for brevity. The general solution to the unperturbed equations (67) will be taken as:

$$x_0 = a_1 \cos\frac{z}{2} + a_2 \sin\frac{z}{2}, \qquad y_0 = b_1 \cos z + b_2 \sin z. \tag{70}$$

Since the system (6) is autonomous, we may choose the phase of the periodic motion arbitrarily, and we select $a_2 = 0$. Substitution of (70) into (68), (69) leads to the removal of secular terms by the following choice of parameters:

$$\omega_1 = -\frac{a_1^2}{16}, \qquad c_1 = -\frac{a_1^2}{32} \qquad (a_2 = 0). \tag{71}$$

Substitution of (70) and (71) into (68), (69) yields a particular solution for x_1 and y_1 which we omit for brevity. Substituting it into the x_2, y_2 equations and removing secular terms requires either that $b_1 = 0$ or $b_2 = 0$. Each of these cases gives a distinct periodic motion. In each case, the removal of secular terms gives values for ω_2 and c_2:

$$b_1 = 0\colon \omega_2 = \frac{a_1^2}{768}[26b_2^2 + 5a_1^2], \quad c_2 = \frac{13a_1^4 - 4b_2^2[a_1^2 + 26b_2^2]}{3072}, \tag{72}$$

$$b_2 = 0\colon \omega_2 = \frac{a_1^2}{384}[13b_1^2 + a_1^2], \quad c_2 = \frac{7a_1^4 + 4b_1^2[11a_1^2 - 26b_1^2]}{3072}. \tag{73}$$

In order to show that these two families of periodic motions are born as one crosses the transition curves (17), (18), we need to compute an expression for the energy h of equation (7). In the case $b_1 = 0$ we obtain:

$$h = \frac{a_1^2}{8} + \frac{b_2^2}{2} + \varepsilon a_1^2\left[\frac{2b_2^2 - a_1^2}{64}\right] + \varepsilon^2 a_1^2\left[\frac{53b_2^4 + 152a_1^2b_2^2 + 78a_1^4}{36864}\right] + \cdots, \tag{74}$$

while in the case $b_2 = 0$ the comparable expression is:

$$h = \frac{a_1^2}{8} + \frac{b_1^2}{2} + \varepsilon a_1^2\left[\frac{2b_1^2 - a_1^2}{64}\right] + \varepsilon^2 a_1^2\left[\frac{53b_1^4 + 440a_1^2b_1^2 + 42a_1^4}{36864}\right] + \cdots. \tag{75}$$

Note that when $b_1 = b_2 = 0$, the derived periodic motion (70) reduces to the torsional x-mode. Thus each of the periodic motions to have derived bifurcates out of the x-mode as the energy h increases through a critical value. In the case of $b_1 = 0$ this is obtained from (74) as

$$h = \frac{a_1^2}{8} - \varepsilon\frac{a_1^4}{64} + \varepsilon^2\frac{78a_1^4}{36864} + \cdots, \qquad b_1 = 0. \tag{76}$$

On the other hand, equations (66), (71), (72) give for this case,

$$p^2 = \frac{1}{4} - \varepsilon \frac{a_1^2}{32} + \varepsilon^2 \frac{13a_1^4}{3072} + \cdots, \qquad b_1 = 0. \tag{77}$$

In order to compare this with the transition curves (17), (18), we write

$$p^2 = \frac{1}{4} + \nu_1 E + \nu_2 E^2 + \cdots, \tag{78}$$

where $E = \varepsilon h$, and substitute (76), (77) into (78), collect terms, and solve for the ν_i, giving:

$$p^2 = \frac{1}{4} - \frac{1}{4} E + \frac{1}{48} E^2 + \cdots, \tag{79}$$

which agrees with the transition curve (18). Similar treatment of the case $b_2 = 0$ gives the transition curve (17).

Thus we have shown that a periodic motion of the approximate form

$$x = a_1 \cos \frac{\omega t}{2}, \qquad y = b_2 \sin \omega t, \tag{80}$$

where

$$\omega = 1 - \frac{a_1^2}{16} \varepsilon + \frac{5}{768} a_1^4 \varepsilon^2 + \cdots, \tag{81}$$

bifurcates out of the x-mode as the latter changes its stability as one crosses the transition curve (18). Since b_2 may be either positive or negative, two such modes are born together in a pitchfork bifurcation. Similarly a pair of periodic motions of the approximate form

$$x = a_1 \cos \frac{\omega t}{2}, \qquad y = b_1 \cos \omega t, \tag{82}$$

where

$$\omega = 1 - \frac{a_1^2}{16} \varepsilon + \frac{1}{384} a_1^4 \varepsilon^2 + \cdots, \tag{83}$$

are born as one crosses the transition curve (17).

Although these motions involve a 2:1 ratio between the frequencies of y- and x-motions, these are not the non-local mode observed by Cusumano, as explained above.

Conclusions

We have investigated the existence, stability and bifurcation of periodic motions in Cusumano's unforced conservative two degree of freedom system S of Figure 2. Specifically, we have obtained approximate expressions for transition curves along which the simple and exact solutions called the torsional x-mode and the bending y-mode lose their stability, and additionally, along which more complicated periodic motions are born. The nature of these bifurcating periodic motions are diverse. In the neighborhood of $p = 1$, i.e., of 1:1 resonance between the x- and y-modes, we found two distinct kinds of bifurcating periodic motions, called nonlinear normal modes and elliptic orbits. In the neighborhood of the large-p transition curve of the x-mode, we found a kind of periodic motion called non-local mode which apparently corresponds to a motion observed by Cusumano in his experiments. This motion involved a 2:1 frequency ratio between y- and x-motions, and was found to exist only for sufficiently large energies. In contrast to the non-local mode, we found yet another type of periodic motion which bifurcates off of the transition curves near $p = 1/2$. Although these periodic motions also involve a 2:1 frequency ratio between y- and x-motions, they exist for arbitrarily small energies, and hence are distinct from the non-local mode.

In all cases, the bifurcating modes occurred via pitchfork bifurcations [2], a reflection of the symmetry of the kinetic and potential energies (1), (2). The modest inquiries which we have presented here are, of course, but the tip of the iceberg of the interconnection of families of periodic motions which are expected to be dense in the phase space of initial conditions ("Poincaré's conjecture" [2]). Other bifurcations are expected to include those associated with the other transition curves along which the x- and y-modes lose their stability (of which there are an infinite number), as well as secondary bifurcations in which the periodic motions which we have investigated lose their stability and give rise to yet more complicated periodic motions.

Although we have not presented an analysis of the stability of the non-local mode, we conjecture that it is stable when it first bifurcates off of the x-mode. This is based on the apparent pitchfork bifurcation through which the non-local mode is born and the accompanying loss of stability of the non-local mode via Floquet theory and perturbations.

It is interesting to contrast the two methods which we used for obtaining the transition curves of Figure 4. On the one hand these may be obtained by a linear analysis which, however, involves d.e.'s with periodic coefficients (Floquet theory). Alternately, we may obtain them by a nonlinear analysis which involves constant coefficient nonlinear d.e.'s (e.g., the slow flow associated with the two-variable expansion method.) These two approaches underline the dual nature of the transition curves, namely that they are both the point of stability change for the primary periodic motion, and they are the point of birth of the bifurcating periodic motions.

The appearance of the non-local mode in this problem is certainly a result of the inertial coupling nonlinearity, i.e., the term $\varepsilon y^2\dot{x}^2/2$ in the kinetic energy, cf. equation (4), since without this term the system is linear. Moreover, comparable systems which involve only nonlinearities in the potential energy, e.g., corresponding to nonlinear spring terms, have not displayed motions comparable to the non-local mode (see, e.g., [9, 10]). The question arises as to the robustness of the non-local mode when the system is changed somewhat. Although we have no definitive answer to this question, we have shown that the non-local mode continues to exist when the following terms are added to the Lagrangian (4):

(a) $\varepsilon x^2\dot{x}^2/2$

(b) $\varepsilon x^2\dot{y}^2/2$

(c) $\varepsilon(\alpha x^4 + \beta x^2 y^2 + \gamma y^4)$

Here (a) and (b) are additional inertial nonlinearities, while (c) is a nonlinear stiffness.

We close by noting that the system S studied in this work also exhibits the phenomenon of chaos. Since S is conservative, chaos occurs for small nonlinearities ε via the process known as KAM theory [2, 4]. The gradual appearance of chaos as ε is increased from zero is closely related to the structure of periodic motions, to their resonance and bifurcations, and especially to the appearance of saddles in the Poincaré map associated with unstable periodic motions.

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Appendix: Asymptotic Expressions for Transition Curves

As explained in the text, we used computer algebra (see [7, 8]) to obtain the following asymptotic expansions for the transition curves governing the stability of the x- and y-modes.

Stability of x-mode, $N = 0$ curve:

$$E = \frac{1}{8p^2} + \frac{1}{32p^4} + \frac{13}{2048p^6} + \frac{7}{8192p^8} - \frac{5}{294912p^{10}} - \frac{511}{9437184p^{12}} - \frac{90431}{4831838208p^{14}} - \frac{64507}{19327352832p^{16}} + \frac{301777}{3865470566400p^{18}} + \frac{53384669}{185542587187200p^{20}} + \frac{1101086183}{10019299708108800p^{22}} + 1 . \tag{A-1}$$

Stability of x-mode, $N = 1$ (first of two):

$$p^2 = \frac{278376506777003E^{12}}{35909170153861939200} + \frac{469375712483E^{11}}{80154397664870400} - \frac{4491673793E^{10}}{333976656936960} - \frac{57943529E^9}{85899345200} - \frac{2902619E^8}{18119393280} - \frac{271897E^7}{1207959552} + \frac{175E^6}{37748736} + \frac{1451E^5}{1179648} + \frac{121E^4}{24576} + \frac{7E^3}{512} + \frac{E^2}{32} - \frac{E}{2} + 1 . \tag{A-2}$$

Stability of $x-$mode, $N = 1$ (second of two):

$$p^2 = \frac{44057341063512097963E^{12}}{35909170153861939200} + \frac{76078873825940029E^{11}}{80154397664870400} + \frac{1182047378608571E^{10}}{1669883284684800} + \frac{11893471795603E^9}{23192823398400} + \frac{6571091641E^8}{1811939280} + \frac{303440089E^7}{1207959552} + \frac{6448559E^6}{37748736} + \frac{134773E^5}{1179648} + \frac{1849E^4}{24576} + \frac{25E^3}{512} + \frac{E^2}{32} - \frac{3E}{2} + 1 . \tag{A-3}$$

Stability of x-mode, $N = 2$ (first of two):

$$p^2 = \frac{764765105807608307E^{12}}{38834570841292800} + \frac{7482367897673E^{11}}{6421059993600} + \frac{7631316587893E^{10}}{11557907988480} + \frac{9964462607E^9}{28665446400} + \frac{4486747001E^8}{28665446400} + \frac{163519E^7}{3981312} - \frac{556481E^6}{19906560} - \frac{317E^5}{4608} - \frac{1277E^4}{13824} - \frac{5E^3}{48} - \frac{5E^2}{48} - \frac{E}{4} + \frac{1}{4} . \tag{A-4}$$

Stability of x-mode, $N = 2$ (second of two):

$$p^2 = -\frac{2195569999772173E^{12}}{388345708412928000} - \frac{12799524341E^{11}}{1284211998720} - \frac{645807356617E^{10}}{57789539942400} - \frac{261373297E^9}{28665446400} - \frac{125403271E^8}{28665446400} + \frac{7873E^7}{3981312} + \frac{172609E^6}{19906560} + \frac{67E^5}{4608} + \frac{259E^4}{13824} + \frac{E^3}{48} + \frac{E^2}{48} - \frac{E}{4} + \frac{1}{4}. \quad \text{(A-5)}$$

Stability of y-mode, $N = 1$ (first of two):

$$p^2 = \frac{188580177E^7}{134217728} - \frac{3658311E^6}{4194304} + \frac{74061E^5}{131072} - \frac{3195E^4}{8192} + \frac{153E^3}{512} - \frac{9E^2}{32} + \frac{E}{2} + 1. \quad \text{(A-6)}$$

Stability of y-mode, $N = 1$ (second of two):

$$p^2 = \frac{135608175E^7}{134217728} - \frac{2725191E^6}{4194304} + \frac{57555E^5}{131072} - \frac{2619E^4}{8192} + \frac{135E^3}{512} - \frac{9E^2}{32} + \frac{3E}{2} + 1. \quad \text{(A-7)}$$

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