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USING COMPUTER ALGEBRA TO HANDLE ELLIPTIC FUNCTIONS IN THE METHOD OF AVERAGING

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ABSTRACT

In this work we apply the method of averaging to the equation

$$\ddot{x} + x + x^2 = \epsilon\ g(x,\dot{x}), \quad \epsilon \ll 1$$

In particular we investigate the occurrence of limit cycles when $g(x,\dot{x})$ is a polynomial.

This involves perturbing off of the Jacobian elliptic function solutions of the equation

$$\ddot{x} + x + x^2 = 0$$

Computer algebra (MACSYMA) programs which perform the necessary computations are presented.

An example is treated and results are compared to those obtained by numerical integration and by traditional trigonometric averaging.

INTRODUCTION

We shall investigate the dynamics of the autonomous system:

$$\ddot{x} + x + x^2 = \epsilon\ g(x,\dot{x}), \quad \epsilon \ll 1 \tag{1}$$

where $g(x,\dot{x})$ is an arbitrary polynomial in x and $\dot{x}$. Eq.1 represents a strongly nonlinear oscillator which is influenced by a small time-independent perturbation $\epsilon\ g(x,\dot{x})$. In particular, we shall examine the limit cycles exhibited by Eq.1. (A limit cycle is a periodic motion which, when represented as a closed orbit in the x-$\dot{x}$ phase plane, is the only closed orbit in its neighborhood.) We note that the apparently more general system:

$$\ddot{x} + \alpha\ x + \beta\ x^2 = \epsilon\ g(x,\dot{x}) \tag{2}$$

where α and β are constants (not necessarily positive), can be reduced to Eq.1 by affine transformations on x and t.

We approach Eq.1 by using a perturbation method (averaging) valid for small ϵ. Since the solution to Eq.1 when $\epsilon = 0$ involves Jacobian elliptic functions (sn,cn, and dn), our computations involve the algebraic manipulation of elliptic functions. We have found that the use of computer algebra (MACSYMA) has reduced the toil and increased the speed and accuracy of these computations.

RELATION TO PREVIOUS WORK

This work represents a companion to previous work on the related system (Ref.3):

$$\ddot{x} + \alpha x + \beta x^3 = \epsilon g(x,\dot{x}) \tag{3}$$

Eq.3 has been generalized to permit α and β to be slowly varying functions of time (Ref.4), and a particular case in which $\alpha = -\cos \epsilon t$, $\beta = 1$ and $g = 0$ has been shown to exhibit chaos (Ref.5). The reader is referred to the Ph.D. thesis of Coppola (Ref.2) for additional work on Eq.3.

ELLIPTIC FUNCTIONS

Since some readers may be unfamiliar with Jacobian elliptic functions, we offer the following brief introduction (see Ref.1). The functions cn and sn are the elliptic counterparts of the trigonometric functions cos and sin. Like cos, cn is even, while like sin, sn is odd. The identity

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1 \tag{4}$$

reminds us of the corresponding relation between sin and cos. The functions cn and sn are actually a family of functions which are parameterized by a "square-modulus" m:

$$\mathrm{cn} = \mathrm{cn}(u,m), \qquad \mathrm{sn} = \mathrm{sn}(u,m) \tag{5}$$

In fact, cn and sn reduce to cos and sin for $m = 0$. The derivatives of cn and sn, however, introduce a new function, dn, which has no trigonometric counterpart:

$$\frac{d}{du}\,\mathrm{cn} = -\,\mathrm{sn}\,\mathrm{dn}, \qquad \frac{d}{du}\,\mathrm{sn} = \mathrm{cn}\,\mathrm{dn} \tag{6}$$

In the trigonometric limit $m = 0$, dn reduces to unity. The function dn satisfies the following equations:

$$m\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1, \qquad \frac{d}{du}\,\mathrm{dn} = -\,m\,\mathrm{sn}\,\mathrm{cn} \tag{7}$$

The functions cn and sn are periodic with period 4K(m), where K(m) is a tabulated function called the complete elliptic integral of the first kind. For m going from 0 to 1, K(m) goes from $\pi/2$ to infinity. The function dn has period 2K(m).

A related quantity is E(m), the complete elliptic integral of the second kind. For m going from 0 to 1, E(m) goes from $\pi/2$ to 1. E(m) enters this work through the integral of cn^2 over one period 4K(m):

$$\int_0^{4K} \mathrm{cn}^2\,du = \frac{4}{m}\,[E(m) - (1-m)\,K(m)]$$

THE UNPERTURBED PROBLEM

For $\epsilon = 0$, Eq.1 takes the form:

$$\ddot{x} + x + x^2 = 0 \tag{8}$$

Although the general solution to Eq.8 may be found by utilizing the conservation of energy, evaluating the resulting elliptic integral and inverting, we find it more instructive to proceed as follows: Assume a solution in the form:

$$x = a_1 + a_2 \, sn^2, \qquad u = \omega t + b \tag{9}$$

where $sn = sn(u,m)$ and where a_1, a_2, ω, b and m are constants. Differentiating Eq.9 and using Eq.6 gives

$$\dot{x} = 2 \, a_2 \, \dot{u} \, sn \, sn' = 2 \, a_2 \, \omega \, sn \, cn \, dn \tag{10}$$

where ' represents differentiation with respect to u. Differentiating Eq.10 and using Eqs.4-7 gives

$$\ddot{x} = 2 \, a_2 \, \omega^2 \, [sn' cn \, dn + sn \, cn' dn + sn \, cn \, dn']$$

$$= 2 \, a_2 \, \omega^2 \, [cn^2 \, dn^2 - sn^2 \, dn^2 - m \, sn^2 \, cn^2]$$

$$= 2 \, a_2 \, \omega^2 \, [(m-1) + 2(1-2m) \, cn^2 + 3m \, cn^4] \tag{11}$$

Substituting Eq.9 and Eq.11 into Eq.8, then using Eq.4 gives

$$k_1 + k_2 \, cn^2 + k_3 \, cn^4 = 0 \tag{12}$$

where $k_1 = 2 \, a_2 \, \omega^2 \, (m-1) + (a_1+a_2) \, (1+a_1+a_2)$

$k_2 = 4 \, a_2 \, \omega^2 \, (1-2m) - a_2 \, (1 + 2(a_1+a_2))$

$k_3 = 6 \, a_2 \, \omega^2 \, m + a_2^2$

Requiring k_1, k_2 and k_3 to vanish gives three nonlinear algebraic equations relating the four parameters a_1, a_2, ω and m. We used MACSYMA to solve these for a_1, a_2 and ω in terms of m (see Appendix I) and obtained:

$$a_1 = \frac{1 + m - \sqrt{\lambda}}{2 \sqrt{\lambda}} \tag{13}$$

$$a_2 = - \frac{3 \, m}{2 \sqrt{\lambda}} \tag{14}$$

$$\omega = \frac{1}{2 \, \lambda^{1/4}} \tag{15}$$

where

$$\lambda = m^2 - m + 1 \tag{16}$$

Thus, the general solution (Eqs.9 and 10) to Eq.8 involves the two arbitrary constants m and b. Since Eq.8 is autonomous, the general solution must be invariant under an arbitrary translation in t, and the arbitrariness of the constant b makes this so. The constant m, on the other hand, determines which integral curve we are on in the x-$\dot{x}$ phase plane.

Since the unperturbed system, Eq.8, is conservative, it admits the energy integral:

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{x^3}{3} = h \tag{17}$$

Substituting Eqs.9 and 10 into Eq.17 and using Eqs.4,7 and 13-16 gives the following relation between energy h and square-modulus m:

$$h = \frac{1}{12}\left[1 - \frac{(m-2)(m+1)(2m-1)}{2\,\lambda^{3/2}}\right] \tag{18}$$

The unperturbed system, Eq.8, has two equilibria in the x-$\dot{x}$ phase plane, a center at the origin and a saddle at $x = -1, \dot{x} = 0$. From Eq.17, the center at the origin corresponds to $h = 0$, which from Eq.18 gives $m = 0$. Similarly, the separatrix which passes through the saddle corresponds to $h = 1/6$ and to $m = 1$. See Fig.1. Thus, each of the integral curves which lie inside the separatrix can be parameterized by a value of m in the interval (0,1). Those trajectories which lie outside the separatrix are not periodic and do not concern us in this study of limit cycles.

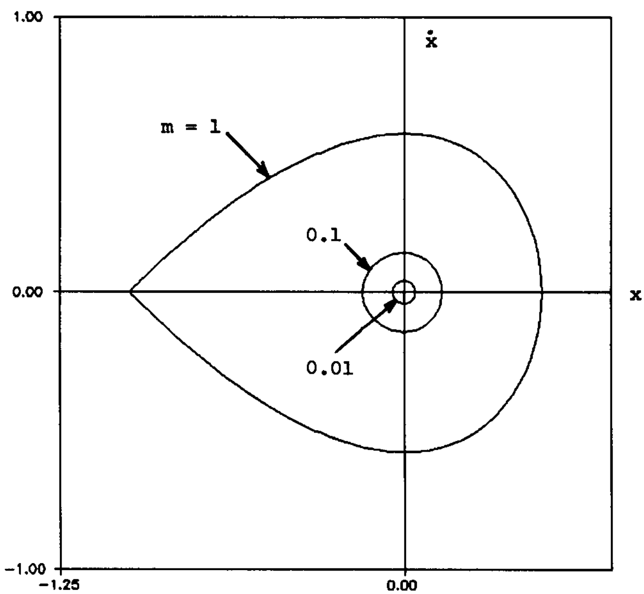

Fig.1. Closed integral curves in the x-$\dot{x}$ phase plane for the unperturbed problem, Eq.8, $\ddot{x} + x + x^2 = 0$. Integral curves corresponding to values of m = 1,0.1 and 0.01 are displayed. The m = 1 curve is the separatrix which passes through the saddle at $\dot{x} = 0$, $x = -1$. The center at the origin corresponds to m = 0.

VARIATION OF PARAMETERS

The general solution of Eq.8, namely Eqs.9 and 10 with parameters given by Eqs.13-16, will be the starting point for our perturbation approximation of Eq.1. We begin by using variation of parameters to express the effect of the order ϵ terms on the slow evolution of the square-modulus m, now considered a function of t. In contrast to the method of averaging, the computations presented in this section are exact. However, the results are intractable and unenlightening. The method of averaging (introduced in the next section) replaces the results obtained in this section by more useful equations, which are, however, approximate (valid in the small ϵ limit.)

In Eq.9, m and b are arbitrary constants of integration. As usual in the method of variation of parameters, we look for a solution to Eq.1 in the form of Eq.9, where the two arbitrary constants m and b are allowed to vary in time. This results in first-order differential equations on m(t) and b(t). For our application to the problem of limit cycles, we shall consider only the equation on m(t), which may be found as follows:

Let us write the general solution, Eq.9, in the abstract form:

$$x = x(t,m,b) \tag{19}$$

Differentiating Eq.19 gives

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial m}\frac{dm}{dt} + \frac{\partial x}{\partial b}\frac{db}{dt} \tag{20}$$

As usual in variation of parameters, we require that $\dot{x}$ be given by Eq.10, i.e.,

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} \tag{21}$$

giving

$$\frac{\partial x}{\partial m}\frac{dm}{dt} + \frac{\partial x}{\partial b}\frac{db}{dt} = 0 \tag{22}$$

Differentiating Eq.21 gives

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 x}{\partial m\partial t}\frac{dm}{dt} + \frac{\partial^2 x}{\partial b\partial t}\frac{db}{dt} \tag{23}$$

in which the unperturbed solution satisfies Eq.8,

$$\frac{\partial^2 x}{\partial t^2} + x + x^2 = 0 \tag{24}$$

and where

$$\frac{d^2x}{dt^2} + x + x^2 = \epsilon\, g \tag{25}$$

Substituting Eqs.24 and 25 into Eq.23 gives

$$\frac{\partial^2 x}{\partial m\partial t}\frac{dm}{dt} + \frac{\partial^2 x}{\partial b\partial t}\frac{db}{dt} = \epsilon\, g \tag{26}$$

Eqs.22 and 26 may be solved simultaneously for $\dot{m}$ as follows:

$$\dot{m} = \frac{\epsilon\, g\, x_b}{x_{mt}x_b - x_{bt}x_m} \tag{27}$$

where subscripts represent partial differentiation.

We used MACSYMA to substitute Eq.9 into Eq.27, and then to simplify the result by using various identities and the parameter values Eqs.13-16. See Appendix II, where we obtained:

$$\dot{m} = -\frac{8}{3}\frac{\lambda^{7/4}}{1-m} \text{ sn cn dn } \epsilon\, g' \tag{28}$$

where $g = g(x,\dot{x}) = g(a_1+a_2 \text{ sn}^2, 2a_2\omega \text{ sn cn dn})$.

THE METHOD OF AVERAGING

The right hand side (rhs) of Eq.28 is periodic in t and thus in the appropriate form for averaging. (See Ref.8 for a thorough treatment of averaging.) The averaged version of Eq.28 is obtained by replacing the rhs of Eq.28 by its average value taken over one period of length 4K(m):

$$\dot{m} = -\frac{8}{3}\frac{\lambda^{7/4}}{1-m}\,\epsilon\,\frac{1}{4K}\int_0^{4K} \text{sn cn dn } g \, du \tag{29}$$

where $g = g(a_1+a_2 \text{ sn}^2, 2a_2\omega \text{ sn cn dn})$ and sn = sn(u,m), cn = cn(u,m) and dn = dn(u,m).

If $g(x,\dot{x})$ is a polynomial in x and $\dot{x}$, then the evaluation of the integral in Eq.29 may be readily accomplished. Terms in g of the form $x^N \dot{x}^M$ for M even lead to integrals of the form

$$\int_0^{4K} \text{cn}^P \text{ sn dn } du \tag{30}$$

where P is an integer, which vanish due to the oddness of the integrand. On the other hand, terms in g of the form $x^N \dot{x}^M$ for M odd lead to integrals of the form

$$\int_0^{4K} \text{cn}^{2P} \, du \tag{31}$$

which may be evaluated by using the following results from Ref.1, pp.192-3, formulas 312: Define

$$C_{2P} = \int_0^{4K} \text{cn}^{2P} \, du \tag{32}$$

Then

$$C_0 = 4K, \quad C_2 = \frac{4}{m}[E-(1-m)K] \tag{33}$$

and

$$C_{2P+2} = \frac{2P}{2P+1}\frac{2m-1}{m} C_{2P} + \frac{2P-1}{2P+1}\frac{1-m}{m} C_{2P-2} \tag{34}$$

where E = E(m) and K = K(m) are complete elliptic integrals.

An equilibrium point of the averaged $\dot{m}$ equation, Eq.29, corresponds to a limit cycle in the original equation (Eq.1). Thus if $m = m_0$ is a root of the equation

$$\int_0^{4K} \mathrm{sn}\ \mathrm{cn}\ \mathrm{dn}\ g\ du = 0 \tag{35}$$

then the averaged equation predicts that for small ϵ, a limit cycle coincides with the energy curve, Eq.17, associated with a value of $h = h_0$ which corresponds to m_0 by Eq.18.

For an arbitrary polynomial $g(x,\dot{x})$, the limit cycle integral condition, Eq.35, may be efficiently evaluated by using computer algebra. See Appendix III where we present a MACSYMA program which performs the integration in Eq.35 by using Eqs.33 and 34.

EXAMPLE

As an example let us take

$$\ddot{x} + x + x^2 = \epsilon\ (k_{01}\ \dot{x} + k_{11}\ x\ \dot{x}) \tag{36}$$

in which k_{01} and k_{11} are given parameters. As shown in Appendix III, the limit cycle integral condition, Eq.35, becomes in the case of Eq.36:

$$\frac{k_{01}}{k_{11}} = \frac{1}{2} + \frac{5}{14}\ \frac{(K-2E)m^3+(K+3E)m^2+(-4K+3E)m+2(K-E)}{\sqrt{\lambda}\ \ [(K-2E)m^2+(-3K+2E)m+2(K-E)]} \tag{37}$$

where λ is given by Eq.16 and where K and E are complete elliptic integrals.

For $m = 0$, Eq.37 is indeterminate. However, its limiting value may be determined by expressing K and E as power series in m. The following series are from Ref.1, pp.297-8, formulas 900:

$$K(m) = \frac{\pi}{2}\left[1 + \frac{1}{4}\,m + \frac{9}{64}\,m^2 + \cdots\right] \tag{38}$$

$$E(m) = \frac{\pi}{2}\left[1 - \frac{1}{4}\,m - \frac{3}{64}\,m^2 + \cdots\right] \tag{39}$$

Substituting Eqs.38 and 39 into Eq.37 gives $k_{01}/k_{11} \to 0$ as $m \to 0$.

Eq.37 is also indeterminate for $m = 1$. In the limit as $m \to 1$, K(m) and E(m) have the following leading behaviors (Ref.1, p.298, formulas 900):

$$K(m) = \frac{1}{2}\ln\frac{16}{1-m} + \cdots, \qquad E(m) = 1 + \cdots \tag{40}$$

Substituting Eq.40 into Eq.37 gives $k_{01}/k_{11} \to \frac{1}{2} - \frac{5}{14} = \frac{1}{7}$ as $m \to 1$.

Fig.2 shows a graph of Eq.37. Note that averaging predicts no limit cycle if $k_{01}/k_{11} > 1/7$ or if $k_{01}/k_{11} < 0$, and a single limit cycle for $0 < k_{01}/k_{11} < 1/7$.

In order to check this result, we numerically integrated Eq.36 using a Runge-Kutta scheme and found excellent agreement with the averaging predictions. See Fig.3.

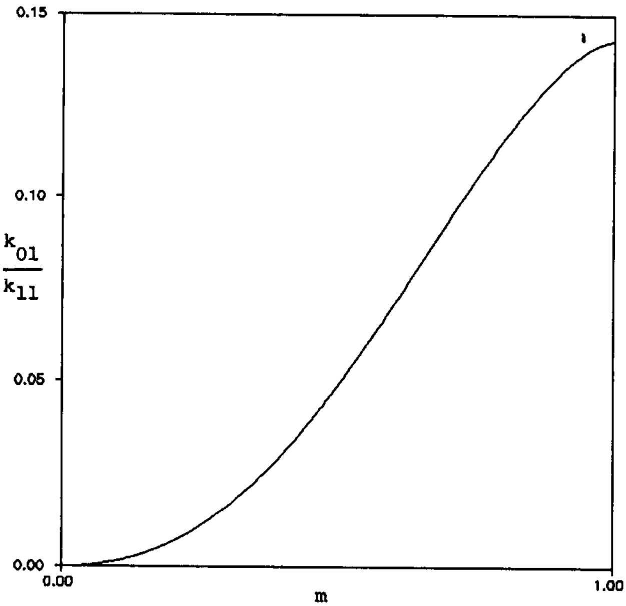

Fig.2. Graph of Eq.37 showing the relationship between the ratio of coefficients k_{01}/k_{11} in Eq.36 and the square-modulus m. Here m corresponds to the integral curve of the unperturbed equation (Eq.8) which is predicted to be a limit cycle for Eq.36.

COMPARISON WITH TRIGONOMETRIC AVERAGING

In order to offer a comparison of elliptic averaging with traditional trigonometric averaging, we treat the previous example by trigonometric averaging.

In order to write Eq.36 in a form which is suitable for trigonometric averaging, the quadratic terms are taken to be $O(\epsilon)$, while the $\dot{x}$ term is taken to be $O(\epsilon^2)$. This choice, which is discussed in Ref.7 (pp.10-13), is motivated by the fact that if the $\dot{x}$ damping term is chosen to be $O(\epsilon)$, no limit cycle results. Thus we take Eq.36 in the form:

$$\ddot{x} + x = \epsilon^2 a \dot{x} + \epsilon b x \dot{x} + \epsilon c x^2 \tag{41}$$

where

$$a = \frac{k_{01}}{\epsilon}, \quad b = k_{11}, \quad c = -\frac{1}{\epsilon} \tag{42}$$

Eq.41 can be treated by averaging using standard MACSYMA programs given in Ref.7 (Chapter 5). The averaging process must be carried out to $O(\epsilon^2)$ (since there are no $O(\epsilon)$ contributions), and the result is as follows:

The approximate solution to Eq.41 is written in the form

$$x = r \sin(t+\theta) \tag{43}$$

where r and θ are slowly varying functions of t. The averaged equation on r is found to be:

$$\dot{r} = \epsilon^2 \left[\frac{ar}{2} + \frac{bcr^3}{8} \right] \tag{44}$$

The equilibria of Eq.44 correspond to limit cycles. Solving for r, we find:

$$r^2 = -\frac{4a}{bc} = 4\frac{k_{01}}{k_{11}} \tag{45}$$

where we have used Eq.42. Since $r^2 > 0$ for real solutions, we see that $O(\epsilon^2)$ trigonometric averaging predicts there to be a limit cycle for $k_{01}/k_{11}>0$. See Fig.3. This result is in disagreement with elliptic averaging for $k_{01}/k_{11}>1/7$, where no limit cycle is predicted. Comparison with numerical integration shows the elliptic averaging to be correct for small ε.

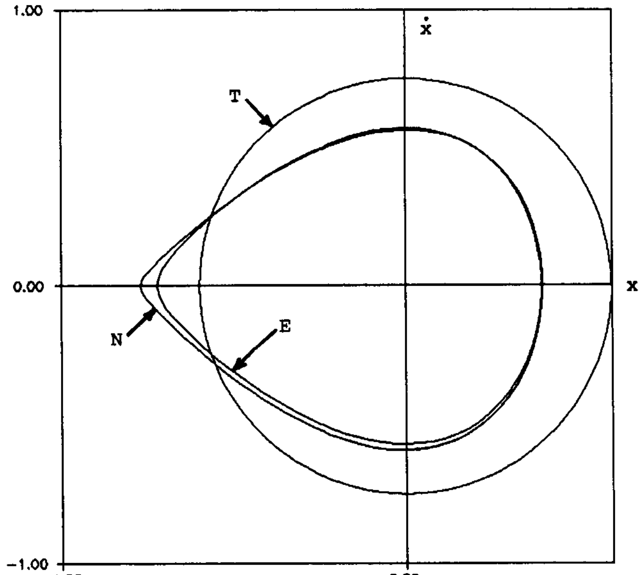

Fig.3. A comparison between elliptic averaging (E), numerical integration (N) and trigonometric averaging (T) approximations for the limit cycle in Eq.36, for parameters $\epsilon = 0.1$, $k_{01} = 0.14$ and $k_{11} = 1$. The elliptic averaging result involves solving Eq.37 for m, which gives m = 0.950577. The integral curve may then be obtained from Eqs.17 and 18. The trigonometric averaging approximation is a circle of radius r = 0.74833 and is obtained from Eq.45. Note that both elliptic and trigonometric averaging approximations are symmetric about the x-axis, while the exact limit cycle (as represented by the numerical integration approximation) is not.

GENERALIZATION

The foregoing treatment of Eq.1 may be easily generalized to a similar system with slowly varying coefficients:

$$\ddot{x} + \alpha(\tau)\, x + \beta(\tau)\, x^2 = \epsilon\, g(x,\dot{x},\tau), \qquad \epsilon \ll 1 \tag{46}$$

where $\dot{}$ represents differentiation with respect to t, and where $\tau = \epsilon t$ represents slow time. In order to treat Eq.46 by averaging, we append to it the auxiliary equation (see Ref.8, p.101):

$$\frac{d\tau}{dt} = \epsilon \tag{47}$$

and treat τ as a dependent variable. Thus in the unperturbed $\epsilon \to 0$ limit, $\dot{\tau}$ vanishes and τ behaves like a constant, as do α and β.

The following results were derived by using MACSYMA programs similar to those in the Appendices, and are presented here for the benefit of those readers who do not have access to MACSYMA:

The solution to Eq.46 may be written in the form:

$$x = a_1 + a_2\, sn^2, \qquad \dot{x} = 2\, a_2\, \omega\, sn\, cn\, dn \tag{48}$$

where sn = sn(u,m), cn=cn(u,m), dn=dn(u,m), and where

$$u = \omega\, t + b \tag{49}$$

$$\omega = \frac{\sqrt{\alpha}}{2\,\lambda^{1/4}}, \qquad \lambda = m^2 - m + 1 \tag{50}$$

$$a_1 = \frac{-\alpha}{2\beta}\,\frac{\sqrt{\lambda} - m - 1}{\sqrt{\lambda}} \tag{51}$$

$$a_2 = -\frac{3\,\alpha\, m}{2\,\beta\,\sqrt{\lambda}} \tag{52}$$

where m satisfies the exact differential equation (obtained by variation of parameters):

$$\frac{dm}{dt} = \epsilon \left[A\,\frac{1}{\alpha}\,\frac{d\alpha}{d\tau} + B\,\frac{1}{\beta}\,\frac{d\beta}{d\tau} + G\, g \right] \tag{53}$$

and where

$$A = \frac{2\,\lambda}{3(m-1)} \left[f_1 + f_2\sqrt{\lambda} \right] \tag{54}$$

$$f_1 = -(m-1)(2m-1) + 2\,\lambda\, cn^2 \tag{55}$$

$$f_2 = 1-m + 2(2m-1)\, cn^2 - 3m\, cn^4 \tag{56}$$

$$B = -A + \frac{2\,m}{m-1}\,\lambda\, cn^2\, sn^2\, dn^2 \tag{57}$$

$$G = \frac{8\,\beta}{3\,\alpha^{3/2}}\,\frac{\lambda^{7/4}}{m-1}\, cn\, sn\, dn \tag{58}$$

The averaged version of Eq.53 is obtained by replacing the rhs of Eq.53 by its average over one period, during which process m,b and τ are held constant. Computer algebra evaluation shows that the averages of the quantities A and B of Eq.53 are proportional to one another:

$$\bar{A} = \frac{1}{4K} \int_0^{4K} A \, du = 5 \, H(m) \tag{59}$$

$$\bar{B} = \frac{1}{4K} \int_0^{4K} B \, du = - 4 \, H(m) \tag{60}$$

where

$$H(m) = \frac{2 \lambda}{15m} \left[2-m + \frac{2 \lambda}{m-1} \frac{E(m)}{K(m)} \right] \tag{61}$$

Thus the averaged version of Eq.53 becomes

$$\frac{dm}{dt} = \epsilon \, H(m) \left[5 \, \frac{1}{\alpha} \frac{d\alpha}{d\tau} - 4 \, \frac{1}{\beta} \frac{d\beta}{d\tau} \right] + \epsilon \, \frac{1}{4K} \int_0^{4K} G \, g \, du \tag{62}$$

which may be written in the form

$$\frac{dm}{dt} = \epsilon \, H(m) \, \frac{d}{d\tau} \, \ell n \left[\frac{\alpha^5}{\beta^4} \right] + \epsilon \, \frac{1}{4K} \int_0^{4K} G \, g \, du \tag{63}$$

In the averaged Eq.63, the integral $\int_0^{4K} G \, g \, du$ is evaluated on the solution, Eq.48. For a particular choice of $g(x,\dot{x},\tau)$, a MACSYMA program similar to that in Appendix III may be used to evaluate this integral. For the special case in which $g \equiv 0$, this integral vanishes and the averaged equation (Eq.63) is separable.

CONCLUSIONS

We have presented a study of the application of averaging to Eq.1. By perturbing off of the elliptic function solutions of Eq.8, we have been able to predict the occurrence and approximate location of limit cycles in Eq.1. In some cases, the approximation based on elliptic functions gives much better results than the traditional approximation based on trigonometric functions. Due to the complicated algebraic manipulations involved in this work, we have found that using computer algebra offers greater convenience, accuracy and efficiency than hand computations. The sample MACSYMA programs included in the Appendices are intended to guide the potential user of computer algebra in the handling of elliptic functions.

It should be noted that the limit cycle problem considered in this work represents perhaps the simplest problem which can be treated using elliptic functions and averaging. Extensions of this work would include the effects of forcing, i.e., nonautonomous systems. As we have shown, the case of slowly varying coefficients is particularly amenable to treatment by averaging. Other important extensions are second-order averaging, i.e., the inclusion of $O(\epsilon^2)$ terms, and consideration of the averaged equation on the phase variable. Because of the great quantity of algebra involved in these topics, computer algebra is an essential tool in their development.

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APPENDIX I

In this Appendix we use MACSYMA to derive Eqs.13-16. (See Refs.6 and 7 for an introduction to MACSYMA.) The following MACSYMA program rederives Eq.12 of the paper, then solves the three simultaneous nonlinear algebraic equations $k_1=0$, $k_2=0$, $k_3=0$. A partial record of the run follows the program listing. MACSYMA returns six solution branches to the three simultaneous equations in Line D19. In order to determine which of the six branches is the appropriate choice, we set m = 0 and require that the origin be a solution, i.e., $a_1=0$, $a_2=0$ and $\omega>0$, see Line D20. Eqs.13-16 are seen to correspond to the third branch in Line D19.

```
depends([cn,sn,dn],[u,m]);
depends(u,[t,w,b]);
depends([a1,a2,w],m);
derivs:[diff(cn,u)=-sn*dn, diff(sn,u)=cn*dn, diff(dn,u)=-m*sn*cn];
idents:[sn=sqrt(1-cn^2), dn= sqrt(1-m+m*cn^2)];
x:a1+a2*sn^2;
xdot:diff(x,t);
xdot:xdot,derivs;
de:diff(xdot,t)+x+x^2;
de:de,derivs;
de:de,idents;
de:expand(de);
de:de,u=w*t+b,diff;
k1:coeff(de,cn,0);
k2:coeff(de,cn,2);
k3:coeff(de,cn,4);
solve([k1,k2,k3],[a1,a2,w]);
%,m=0;
```

Here is a partial record of the run:

```
(C16) k1:coeff(de,cn,0);
                         2          2     2                 2
(D16)          2 A2 M W  - 2 A2 W  + A2  + 2 A1 A2 + A2 + A1  + A1

(C17) k2:coeff(de,cn,2);
                             2           2      2
(D17)               - 8 A2 M W  + 4 A2 W  - 2 A2  - 2 A1 A2 - A2

(C18) k3:coeff(de,cn,4);
                                           2     2
(D18)                              6 A2 M W  + A2

(C19) solve([k1,k2,k3],[a1,a2,w]);

(D19) [[A1 = 0, A2 = 0, W = %R1], [A1 = - 1, A2 = 0, W = %R2],

             2                                           2
     SQRT(M   - M + 1) - M - 1             3 M SQRT(M   - M + 1)
[A1 = - -------------------------, A2 = - ---------------------,
                   2                               2
        2 SQRT(M   - M + 1)                    2 M   - 2 M + 2

                                        2
             1                  SQRT(M   - M + 1) + M + 1
W = -------------------], [A1 = - -------------------------,
       2           1/4                       2
    2 (M   - M + 1)               2 SQRT(M   - M + 1)

                  2
     3 M SQRT(M   - M + 1)                 %I
A2 = ---------------------, W = - -------------------],
             2                       2           1/4
         2 M   - 2 M + 2          2 (M   - M + 1)

             2                                           2
     SQRT(M   - M + 1) - M - 1             3 M SQRT(M   - M + 1)
[A1 = - -------------------------, A2 = - ---------------------,
                   2                               2
        2 SQRT(M   - M + 1)                    2 M   - 2 M + 2

                                          2
               1                  SQRT(M   - M + 1) + M + 1
W = - -------------------], [A1 = - -------------------------,
         2           1/4                       2
      2 (M   - M + 1)               2 SQRT(M   - M + 1)

                  2
     3 M SQRT(M   - M + 1)              %I
A2 = ---------------------, W = -------------------]]
             2                     2           1/4
         2 M   - 2 M + 2        2 (M   - M + 1)

(C20) %,m=0;
(D20) [[A1 = 0, A2 = 0, W = %R1], [A1 = - 1, A2 = 0, W = %R2],

                            1                                 %I
[A1 = 0, A2 = 0, W = -], [A1 = - 1, A2 = 0, W = - --],
                            2                                 2

                              1                               %I
[A1 = 0, A2 = 0, W = - -], [A1 = - 1, A2 = 0, W = --]]
                              2                               2
```

APPENDIX II

In this Appendix we use MACSYMA to evaluate Eq.27:

$$\dot{m} = \frac{\epsilon\, g\, x_b}{x_{mt}x_b - x_{bt}x_m} \tag{27}$$

where subscripts represent partial differentiation.

Eq.27 involves the quantities x_m, x_b, x_{xt}, x_{bt}, which we now consider. We obtain x_m by differentiating Eq.9 with respect to m (while holding t and b fixed):

$$\frac{\partial x}{\partial m} = \frac{\partial a_1}{\partial m} + \frac{\partial a_2}{\partial m}\,sn^2 + 2a_2\,sn\left[\frac{\partial sn}{\partial u}\frac{\partial u}{\partial m} + \frac{\partial sn}{\partial m}\right] \tag{64}$$

in which

$$\frac{\partial u}{\partial m} = \frac{\partial \omega}{\partial m}\,t \tag{65}$$

$$\frac{\partial sn}{\partial u} = cn\ dn \tag{66}$$

$$\frac{\partial sn}{\partial m} = -\ cn\ dn\ f(u,m) \tag{67}$$

where

$$f(u,m) = \frac{E(u)-(1-m)u}{2m(1-m)} - \frac{sn\ cn}{2(1-m)dn} \tag{68}$$

Eq.65 follows from Eqs.9,15 and 16. Eq.66 is equivalent to Eq.6. Eqs.67 and 68 are from Ref.1 (p.283, formulas 710), in which E(u) is the incomplete elliptic integral of the second kind.

The quantity x_{mt} may be similarly computed. Differentiating Eq.10 with respect to m, we find:

$$\frac{\partial^2 x}{\partial m \partial t} = 2\,\frac{\partial a_2}{\partial m}\,\omega\ sn\ cn\ dn + 2a_2\frac{\partial \omega}{\partial m}\,sn\ cn\ dn + 2a_2\omega\ cn\ dn\left[\frac{\partial sn}{\partial u}\frac{\partial u}{\partial m} + \frac{\partial sn}{\partial m}\right]$$

$$+\ 2a_2\omega\ sn\ dn\left[\frac{\partial cn}{\partial u}\frac{\partial u}{\partial m} + \frac{\partial cn}{\partial m}\right] + 2a_2\omega\ sn\ cn\left[\frac{\partial dn}{\partial u}\frac{\partial u}{\partial m} + \frac{\partial dn}{\partial m}\right] \tag{69}$$

where $\frac{\partial cn}{\partial m}$ and $\frac{\partial dn}{\partial m}$ are given by the expressions (cf. Eqs.67 and 68):

$$\frac{\partial cn}{\partial m} = cn\ dn\ f(u,m) \tag{70}$$

$$\frac{\partial dn}{\partial m} = m\ sn\ cn\ f(u,m) - \frac{1}{2}\frac{sn^2}{dn} \tag{71}$$

The quantity x_b is found by differentiating Eq.9 with respect to b:

$$\frac{\partial x}{\partial b} = 2\ a_2\ sn\ cn\ dn \tag{72}$$

Finally, the quantity x_{bt} is obtained by differentiating Eq.10 with respect to b:

$$\frac{\partial^2 x}{\partial b \partial t} = 2\ a_2\ \omega\left[cn^2dn^2 - sn^2dn^2 - m\ sn^2cn^2\right] \tag{73}$$

The following MACSYMA program computes these quantities, substitutes them into Eq.27 and simplifies the result, yielding Eq.28 of the paper:

```
depends([cn,sn,dn],[u,m]);
depends(u,[t,w,b]);
depends([a1,a2,w],m);
derivs:['diff(cn,u)=-sn*dn,'diff(sn,u)=cn*dn,'diff(dn,u)=-m*sn*cn,
    'diff(cn,m)=sn*dn*f,'diff(sn,m)=-cn*dn*f,'diff(dn,m)=m*sn*cn*f-sn^2/(2*dn)];
idents:[sn=sqrt(1-cn^2), dn= sqrt(1-m+m*cn^2)];
x:a1+a2*sn^2;
xt:diff(x,t);
xt:xt,derivs;
xbt:diff(xt,b);
xbt:xbt,derivs;
xb:diff(x,b);
xb:xb,derivs;
xm:diff(x,m);
xm:xm,derivs;
xmt:diff(xt,m);
xmt:xmt,derivs;
dmdt:-xb*e*g/(xm*xbt-xmt*xb);
dmdt:ratsimp(dmdt);
dmdt:dmdt,idents,diff;
dmdt:dmdt,derivs;
dmdt:dmdt,u=w*t+b,diff;
dmdt:dmdt,
a1 = -(sqrt(m^2-m+1)-m-1)/(2*sqrt(m^2-m+1)),
a2 = -3*m/(2*sqrt(m^2-m+1)),w = 1/(2*(m^2-m+1)^(1/4)),diff;
dmdt:factor(ratsimp(dmdt));
```

Here is a record of the final expression produced by the program:

```
(C27) dmdt:factor(ratsimp(dmdt));

                      2                  2                 2          7/4
        8 CN SQRT(1 - CN ) E G SQRT((CN  - 1) M + 1) (M  - M + 1)
(D27)   ----------------------------------------------------------------
                                  3 (M - 1)
```

APPENDIX III

In this Appendix we use MACSYMA to evaluate the limit cycle integral condition, Eq.35. The program listing is followed by a partial record of a sample run based on the following choice for $g(x,\dot{x})$:

$$g(x,\dot{x}) = k_{01}\,\dot{x} + k_{11}\,x\,\dot{x}$$

```
g:read("Enter g(x,y), where y=dx/dt");
/* eliminate even powers of y */
g1:ev(g,y=-y);
g2:(g-g1)/2;
/* form integrand of limit cycle integral, eq.(35) */
integrand:g2*sn*cn*dn,x=a1+a2*sn^2,y=2*a2*w*sn*cn*dn$
/* use identities to express sn,dn in terms of cn */
idents:[sn=sqrt(1-cn^2), dn= sqrt(1-m+m*cn^2)];
integrand1:expand(ev(integrand,idents))$
/* set up rules for evaluating the integral of cn^r */
c[0]:4*k(m);
c[2]:4*(e(m)-(1-m)*k(m))/m;
c[r]:=(r-2)*(2*m-1)/((r-1)*m)*c[r-2]+(r-3)*(1-m)/((r-1)*m)*c[r-4];
/* perform the integration by replacing cn^r by c[r] */
max:hipow(integrand1,cn);
```

```
for i:0 thru max step 2 do b[i]:coeff(integrand1,cn,i);
result:sum(b[2*i]*c[2*i],i,0,max/2)$
/* substitute the parameters a1,a2,w in terms of m */
result1:result,
        a1 = -(sqrt(m^2-m+1)-m-1)/(2*sqrt(m^2-m+1)),
        a2 = -3*m/(2*sqrt(m^2-m+1)), w = 1/(2*(m^2-m+1)^(1/4))$
/* simplify the result */
result2:ratsimp(result1);
```

Here is a partial record of a sample run in which only the first and last lines of the program are displayed:

```
(C3) g:read("Enter g(x,y), where y=dx/dt");
Enter g(x,y), where y=dx/dt
K01*Y+K11*X*Y;
(D3)                K11 X Y + K01 Y

                                        . . .

(C16) /* simplify the result */
result2:ratsimp(result1);

                        2                        2
(D16) - (SQRT(M  - M + 1) ((7 K11 M  - 21 K11 M + 14 K11) K(M)

                         2
 + (- 14 K11 M  + 14 K11 M - 14 K11) E(M))

            2                          2
 + SQRT(M  - M + 1) ((- 14 K01 M  + 42 K01 M - 28 K01) K(M)

                  2
 + (28 K01 M  - 28 K01 M + 28 K01) E(M))

               3               2
 + (5 K11 M  + 5 K11 M  - 20 K11 M + 10 K11) K(M)

                  3                2
 + (- 10 K11 M  + 15 K11 M  + 15 K11 M - 10 K11) E(M))

     2             1/4         3          2
/((M  - M + 1)      (35 M  - 35 M  + 35 M))
```

Finally we obtain Eq.37 of the paper by solving the previous condition for k_{01}:

```
(C18) SOLVE(RESULT2,K01);

                                3            2                        2
(D18) [K01 = ((5 K11 M  + SQRT(M  - M + 1) (7 K11 M  - 21 K11 M + 14 K11)

               2                                              3
 + 5 K11 M  - 20 K11 M + 10 K11) K(M) + (- 10 K11 M

            2                         2                                   2
 + SQRT(M  - M + 1) (- 14 K11 M  + 14 K11 M - 14 K11) + 15 K11 M  + 15 K11 M

                                  2                  2
 - 10 K11) E(M))/(SQRT(M  - M + 1) (14 M  - 42 M + 28) K(M)

               2                         2
 + (- 28 M  + 28 M - 28) SQRT(M  - M + 1) E(M))]
```