Nonlinear Normal Modes in Two-Degree-of-Freedom Systems

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CONSIDER a holonomic, scleronomous conservative system with generalized coordinates x, y.

Suppose the quadratic terms in the potential energy V are diagonalized (i.e., take x and y as normal coordinates for the linear problem).

Consider a class of systems for which

$$V = ax^2 + by^2 + \alpha x^4 + \beta x^3 y + \gamma x^2 y^2 + \sigma x y^3 + \tau y^4 \quad (1)$$

where a, b, α, β, γ, σ, τ are constants such that V is positive definite and for which the kinetic energy T is of the form

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

Then

$$T + V = h$$

where h is a constant equal to the total energy of the system.

As is well known [1–4][2]

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0 \quad (2)$$

where primes denote differentiation with respect to x.

Normal modes of the system will be solutions of (2) which satisfy [2, pp. 162, 172]

$$y(0) = 0 \quad (3)$$

and

$$V_y - y'V_x = 0 \quad \text{when} \quad V = h \quad (4)$$

Since V has been assumed to be symmetric with respect to the origin, the normal modes must satisfy [2, p. 171]

$$y(-x) = -y(x) \quad (5)$$

To satisfy (3) and (5), assume a solution of the form

$$y(x) = C_1 x + C_3 x^3 + C_5 x^5 + \ldots \quad (6)$$

Substituting (6) and (1) into (2) and equating to zero the coefficient of x, find

$$C_3 = \frac{a - b}{6h}(C_1 + C_1^3) \quad (7)$$

Similarly, an expression for C_{2N+1} in terms of C_1 could be obtained by requiring the coefficient of X^{2N-1} to vanish ($N = 1, 2, 3, \ldots$).

The value of C_1 is obtained from (4) as follows. Suppose (X, Y) is the point P of intersection of the modal curve (6) and the curve $V = h$. Then substituting (6) into (1) at P gives h as a function of X. For small h,

$$X = \left(\frac{h}{a + bC_1^2}\right)^{1/2} + 0(h) \quad (8)$$

Substituting (6)–(8) into (4) at P gives for small h,

$$2(b - a)C_1 + X^2[\beta + 2(\gamma - 2\alpha)C_1 + 2(b - 3a)C_3 + 3(\sigma - \beta)C_1^2 + 2(2\tau - \gamma)C_1^3 - \sigma C_1^4] + 0(X^4) = 0 \quad (9)$$

From (9), $C_1 = 0(h)$. Neglecting terms of $0(h^2)$ in (9), find

$$C_1 = \frac{3h\beta}{(a - b)(9a - b)} + 0(h^2) \quad (10)$$

From (7),

$$C_3 = \left(\frac{a - b}{6h}\right) C_1 + 0(h)$$
$$C_3 = \frac{\beta}{2(9a - b)} + 0(h) \quad (11)$$

Substituting (10) and (11) into (6), and noting $x = 0(X) = 0(h^{1/2})$, find

$$y = \frac{3h\beta}{(a - b)(9a - b)}x + \frac{\beta}{2(9a - b)}x^3 + 0(h^{5/2}) \quad (12)$$

Equation (12) represents an approximation to the nonlinear normal mode corresponding to the linear normal mode $y \equiv 0$. To find an approximation to the nonlinear normal mode corresponding to $x \equiv 0$, interchange x and y throughout. The approximation is not valid on $V = h$ since (2) is singular there. The existence of such normal modes has been treated by Pak and Rosenberg [5].

The relation $Y = Y(X)$, which Kauderer [1] refers to as a "grenzkurve," becomes after inserting (8) into (12),

$$Y = \frac{\beta}{2}\frac{(7a - b)}{(a - b)(9a - b)}X^3 + 0(X^5) \quad (13)$$

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References

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4 Knothe, H., "Satellites and Riemannian Geometry," *Celestial Mechanics*, Vol. 1, 1969, p. 45.

5 Pak, C. H., and Rosenberg, R. M., "On the Existence of Normal Mode Vibrations in Nonlinear Systems," *Quarterly of Applied Mathematics*, Vol. 26, No. 3, Oct. 1968, p. 403.

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[2] Numbers in brackets designate References at end of Note.

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