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The Damped Nonlinear Quasiperiodic Mathieu Equation Near 2:2:1 Resonance

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Abstract. We investigate the damped cubic nonlinear quasiperiodic Mathieu equation

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t + \varepsilon\mu \cos \omega t)x + \varepsilon\mu c \frac{dx}{dt} + \varepsilon\mu\gamma x^3 = 0$$

in the vicinity of the principal 2:2:1 resonance. By using a double perturbation method which assumes that both ε and μ are small, we approximate analytical conditions for the existence and bifurcation of nonlinear quasiperiodic motions in the neighborhood of the middle of the principal instability region associated with 2:2:1 resonance. The effect of damping and nonlinearity on the resonant quasiperiodic motions of the quasiperiodic Mathieu equation is also provided. We show that the existence of quasiperiodic solutions does not depend upon the nonlinearity coefficient γ, whereas the amplitude of the associated quasiperiodic motion does depend on γ.

Key words: parametric excitation, quasiperiodic, resonance

1. Introduction

In this paper, we study the effect of damping and cubic nonlinearity on the dynamics of the linear quasiperiodic (QP) Mathieu equation

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t + \varepsilon\mu \cos \omega t)x = 0. \tag{1}$$

in the vicinity of 2:2:1 resonance. The stability and transition curves for this linear undamped QP Mathieu Equation (1) were first studied by Rand and coworkers [1, 2] in the case $\mu = 1$. Various techniques were used successfully to confirm the validity of the obtained transition curves to Equation (1). Figure 1a illustrates these curves in the case $\varepsilon = 0.1$. In order to understand this complicated figure through analytical treatment, Rand et al. [3] applied a double perturbation procedure, which consists of applying two successive perturbation methods, and determined the transition curves of the QP Mathieu Equation (1) in the vicinity of 2:2:1 resonance.

In Rand et al. [3], Equation (1) was studied in the form

$$\frac{d^2x}{dt^2} + \left(\frac{1}{4} + \delta_1\varepsilon + \varepsilon \cos t + \varepsilon\mu \cos(1 + \varepsilon\Delta)t\right)x = 0. \tag{2}$$

where $\delta = 1/4 + \delta_1\varepsilon$ and $\omega = 1 + \varepsilon\Delta$. When ε is small, both drivers in Equation (2) have frequency 1 or close to 1, while the unforced equation has natural frequency 1/2. This represents the 2:2:1 resonance

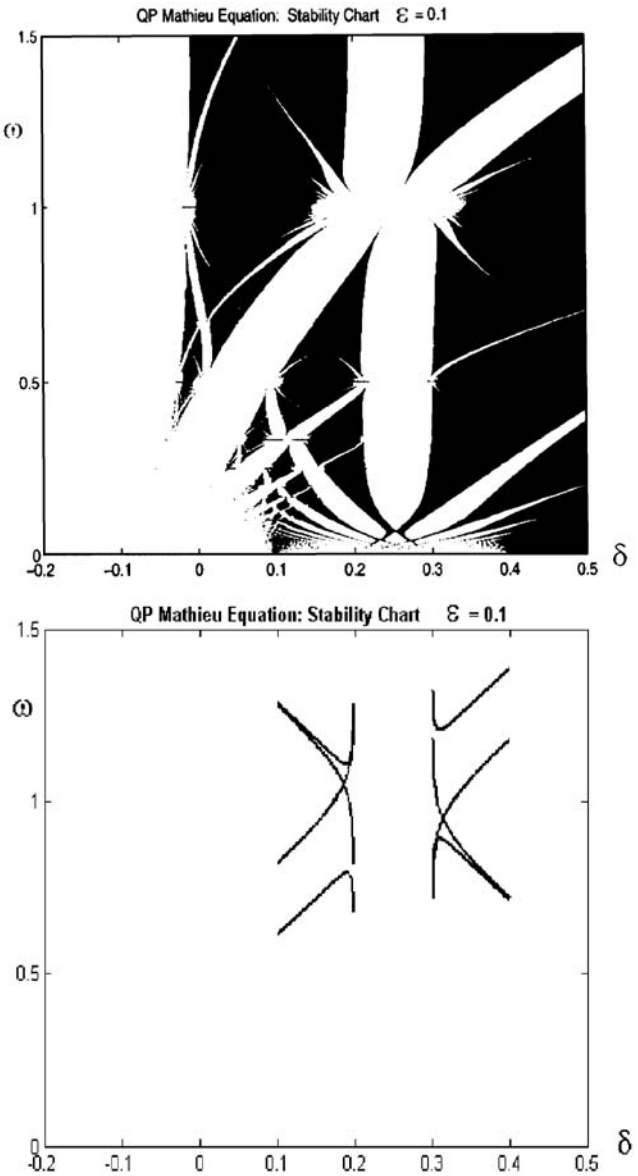

Figure 1. Top (a): Stability of Equation (1) for $\varepsilon = 0.1$, $\mu = 1$, obtained by numerical integration, from [2]. Black: stable, white: unstable. Bottom (b): Perturbation approximation for transition curves separating stable and unstable regions near $\delta = 0.25$, $\omega = 1$ (2:2:1 resonance) from [3].

case. Analytic expressions for the transitions curves separating stable and unstable regions were obtained in Rand et al. [3] by using a double perturbation method valid for small ε and μ. In particular, the analytic approximation of the transition curves bounding the largest of the local instability regions near $\delta = 0.25$ and $\omega = 1$ in Equation (1) were obtained in the form

$$\omega = 1 + \epsilon\left(S \pm \mu\left(\frac{1}{2} + \frac{\delta_1}{S}\right)\right) + O(\epsilon^2) \tag{3}$$

where

$$\delta = \frac{1}{4} + \delta_1 \epsilon \quad \text{and} \quad S = \sqrt{4\delta_1^2 - 1} \tag{4}$$

In Figure 1b, we show these curves for $\varepsilon = 0.1$ and $\mu = 1$.

In the present work, we study the nonlinear damped version of the QP Mathieu Equation (2) in the form

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t + \varepsilon\mu \cos \omega t)x + \varepsilon\mu c \frac{dx}{dt} + \varepsilon\mu\gamma x^3 = 0. \tag{5}$$

where c and γ represent damping and nonlinear component, respectively. Equation (5) can serve, for instance, as a nonlinear damped one-mode model to study convective QP parametric instability in a heated fluid layer vertically forced and thermally driven or to study convective QP gravity modulation instability of a heated fluid layer. For the case of periodic gravity modulation, see [4] and for the periodic thermally driven case, see [5]. The goal here is to investigate the effect of nonlinearity and damping on the existence and bifurcation of QP motions in the vicinity of the principal resonance 2:2:1. In recent works, a similar equation to (5) has been studied. Zounes and Rand [8] investigated the interaction of subharmonic resonance bands and transition from local to global chaos in a cubic undamped nonlinear QP Mathieu equation using Chirikov's overlap criterion [7, 8]. Belhaq et al. [9, 12] developed approaches for generating approximations to solutions of a damped cubic nonlinear QP Mathieu equation. The method of multiple scales was applied twice in order to produce a doubly reduced system whose stationary solutions correspond to QP motions of the original system. More precisely, [9, 12] focused on the approximation of periodic solutions near the generating (principal) resonance point 1:2 to the undamped nonlinear Mathieu equation (unperturbed slow flow). In addition, these approximations were constructed in the vicinity of a stationary solution to the unperturbed autonomous slow flow. This approach provides a local study to QP solutions near a resonant point. In contrast to [9, 12], the contribution of the current paper focuses on the investigation of the effect of nonlinearity and damping on the existence and bifurcation of QP motions in the vicinity of the middle of the largest instability region to the QP Mathieu equation, namely near the principal resonance 2:2:1 corresponding to $\delta = 0.25$ and $\omega = 1$ in Equation (1); see dashed curves in Figure 2. This treatment yields a global analysis for the existence and bifurcation of QP motions near the resonant curve lying in the largest instability region. Note that in Rand et al. [3], analytical approximations to this largest instability region near 2:2:1 resonance were obtained for the linear QP Mathieu Equation (2).

2. Perturbations

We will study Equation (5) in the small ε limit by using the two-variable expansion perturbation method [10] (also known as multiple scales [11]). We replace the independent variable t with two new independent variables, $\xi = t$ and $\eta = \varepsilon t$ ("slow time"). This results in the following expressions for the derivatives of x

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi} + \varepsilon \frac{\partial x}{\partial \eta} \quad \text{and} \quad \frac{d^2x}{dt^2} = \frac{\partial x^2}{\partial \xi^2} + 2\varepsilon \frac{\partial x^2}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial x^2}{\partial \eta^2} \tag{6}$$

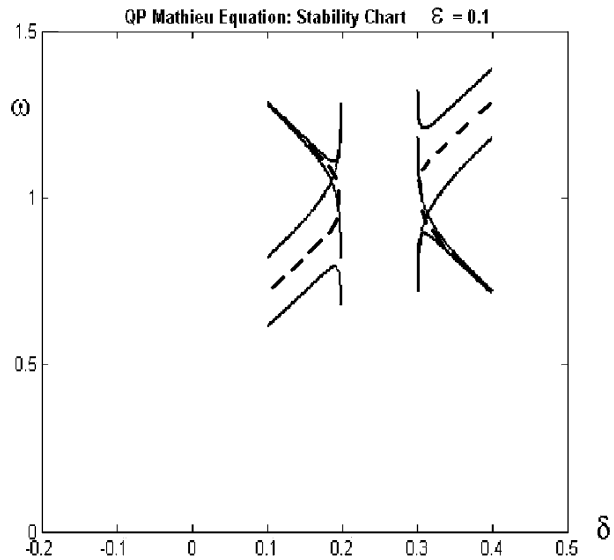

Figure 2. The resonance condition (17) is displayed as a pair of dashed curves which lie in the middle of the instability region shown in Figure 1b. Here $\epsilon = 0.1$ and $\mu = 1$.

We expand x in a power series in ε

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2) \tag{7}$$

and substitute into Equation (5). Then, we collect terms

$$\frac{\partial^2 x_0}{\partial \xi^2} + \frac{1}{4}x_0 = 0 \tag{8}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - \delta_1 x_0 - x_0 \cos\xi - x_0\mu\cos(\xi + \Delta\eta) - c\mu\frac{\partial x_0}{\partial \xi} - \gamma\mu x_0^3 \tag{9}$$

The solution of Equation (8) is given by

$$x_0 = A(\eta)\cos\frac{\xi}{2} + B(\eta)\sin\frac{\xi}{2} \tag{10}$$

Substituting Equation (10) into (9) and removing secular terms gives the following parametric slow flow system

$$\frac{dA}{d\eta} = -\frac{\mu c}{2}A + \left(\delta_1 - \frac{1}{2}\right)B - \frac{\mu}{2}[A\sin(\Delta\eta) + B\cos(\Delta\eta)] + \frac{3}{4}\gamma\mu(A^2 + B^2)B \tag{11}$$

$$\frac{dB}{d\eta} = -\frac{\mu c}{2}B - \left(\delta_1 + \frac{1}{2}\right)A - \frac{\mu}{2}[A\cos(\Delta\eta) - B\sin(\Delta\eta)] - \frac{3}{4}\gamma\mu(A^2 + B^2)A \tag{12}$$

Following [3], we study the slow flow (11), (12) for small μ by using a second perturbation expansion. We expand $A = A_0 + \mu A_1 + O(\mu^2)$ and $B = B_0 + \mu B_1 + O(\mu^2)$. Substituting these into Equations (11),

(12) and collecting terms, we get

$$\frac{dA_0}{d\eta} = B_0\delta_1 - \frac{B_0}{2} \tag{13}$$

$$\frac{dB_0}{d\eta} = -A_0\delta_1 - \frac{A_0}{2} \tag{14}$$

$$\frac{dA_1}{d\eta} = B_1\delta_1 - \frac{B_1}{2} - \frac{dA_0}{d\zeta} + \frac{3B_0^3\gamma}{4} + \frac{3A_0^2B_0\gamma}{4} - \frac{A_0\sin(\Delta\eta)}{2} - \frac{B_0\cos(\Delta\eta)}{2} - \frac{A_0 c}{2} \tag{15}$$

$$\frac{dB_1}{d\eta} = -A_1\delta_1 - \frac{A_1}{2} - \frac{dB_0}{d\zeta} - \frac{3A_0B_0^2\gamma}{4} - \frac{3A_0^3\gamma}{4} + \frac{B_0\sin(\Delta\eta)}{2} - \frac{A_0\cos(\Delta\eta)}{2} - \frac{B_0 c}{2} \tag{16}$$

A_0 and B_0 in Equations (13) and (14) have frequency $\sqrt{\delta_1^2 - 1/4}$. The $\sin\Delta\eta$ and $\cos\Delta\eta$ terms in (15), (16) will be resonant if $\Delta = 2\sqrt{\delta_1^2 - 1/4}$. We consider this case and set

$$\Delta = 2\sqrt{\delta_1^2 - 1/4} \quad \text{or equivalently,} \quad \delta_1 = s\frac{\sqrt{1+\Delta^2}}{2} \tag{17}$$

where $s = \pm 1$.

Equation (17) represents a pair of curves in the δ–ω parameter plane which lie in the middle of the instability region shown in Figure 1b. See Figure 2. Note that in Equation (17), $s = 1$ corresponds to the right branch of the dashed curves in Figure 2, while $s = -1$ corresponds to the left branch. Both branches are included when Equation (17) is written in the form:

$$\Delta^2 = 4\delta_1^2 - 1 \tag{18}$$

In what follows we shall be concerned with the dynamics of Equation (5) in the neighborhood of these curves.

Equations (13) and (14) give

$$A_0 = A_{01}(\zeta)\cos(\Delta\eta/2) + A_{02}(\zeta)\sin(\Delta\eta/2) \tag{19}$$

$$B_0 = \frac{\Delta}{2\delta_1 - 1}\left(A_{02}(\zeta)\cos(\Delta\eta/2) - A_{01}(\zeta)\sin(\Delta\eta/2)\right) \tag{20}$$

Here $A_{01}(\zeta)$ and $A_{02}(\zeta)$ are functions of $\zeta = \mu\eta$. Since $\eta = \varepsilon t$ is slow time, and since we have assumed μ to be small, we shall refer to ζ as "slow slow time", or "s.s. time" for brevity. Substituting (19) and (20) into (15) and (16), and removing secular terms gives the following autonomous s.s. flow system on A_{01} and A_{02}

$$\frac{dA_{01}}{d\zeta} = -\frac{c}{2}A_{01} - \left[\frac{\sqrt{\Delta^2+1}s}{4\Delta} + \frac{1}{4}\right]A_{02} + E\left(A_{01}^2 + A_{02}^2\right)A_{02}, \tag{21}$$

$$\frac{dA_{02}}{d\zeta} = -\frac{c}{2}A_{02} - \left[\frac{\sqrt{\Delta^2+1}s}{4\Delta} + \frac{1}{4}\right]A_{01} - E\left(A_{01}^2 + A_{02}^2\right)A_{01}. \tag{22}$$

where $s = \pm 1$ as in Equation (17) and where

$$E = -\frac{3\gamma}{8\Delta^3}(2\Delta^2 + 3)(s\sqrt{\Delta^2 + 1} + 1) \tag{23}$$

Equations (21) and (22) may be simplified by transforming to polar coordinates, $A_{01} = r\cos\theta$, $A_{02} = r\sin\theta$. This leads to

$$\frac{dr}{d\zeta} = -\frac{c}{2}r - \left[\frac{\sqrt{\Delta^2 + 1}s}{4\Delta} + \frac{1}{4}\right] r\sin 2\theta \tag{24}$$

$$\frac{d\theta}{d\zeta} = -\left[\frac{\sqrt{\Delta^2 + 1}s}{4\Delta} + \frac{1}{4}\right]\cos 2\theta + Er^2 \tag{25}$$

Equilibria in the s.s. flow (24), (25) will correspond to quasiperiodic motions in the original Equation (5). Solving the RHS of (24) for $\sin 2\theta$, we obtain

$$\sin 2\theta = -\frac{2c\Delta}{\sqrt{\Delta^2 + 1}s + \Delta} \tag{26}$$

Bifurcation cases correspond to $\sin 2\theta = \pm 1$. Using Equation (17) the bifurcation cases may be written:

$$2c\Delta = \pm(2\delta_1 + \Delta) \quad \text{or equivalently,} \quad \Delta = \left(\frac{2}{\pm 2c - 1}\right)\delta_1 \tag{27}$$

The last equation in (27) plots as two straight lines through the origin in the δ_1-Δ plane. The corresponding bifurcation points (call them P) correspond to the intersection of these lines with the resonance curves (18) which are depicted as dashed curves in Figure 2. The associated bifurcations are pitchforks of s.s. flow equilibria. In the portions of the resonance curves (18) which lie between the bifurcation points P there exists a pair of nontrivial s.s. flow equilibria, both stable since the origin from which they bifurcate is unstable. See Figure 3 where we show points on the resonance curves (18) which correspond to systems with quasiperiodic motions (nontrivial slow flow equilibria) versus systems whose steady state is only the origin. Note that there is a qualitative difference between $c \leq 1$ versus $c > 1$. When the damping coefficient $c > 1$, only a finite portion of the resonance curve (17) corresponds to nontrivial slow flow equilibria, whereas when $c \leq 1$, the corresponding portion of the resonance curve is infinite. This result is obtained by substituting the last equation in (27) into the resonance curves Equation (18), giving:

$$\Delta^2 = \left(\frac{2}{\pm 2c - 1}\right)^2 \delta_1^2 = 4\delta_1^2 - 1 \tag{28}$$

Solving for δ_1^2, we obtain:

$$\delta_1^2 = \frac{(\pm 2c - 1)^2}{16c(c \pm 1)} \tag{29}$$

For a real value of δ_1, the RHS of (29) must be nonnegative. In the case of the upper sign in (29) there is no restriction on the damping coefficient c (which is assumed to be positive). However, in the case of the lower sign, we must require $c > 1$.

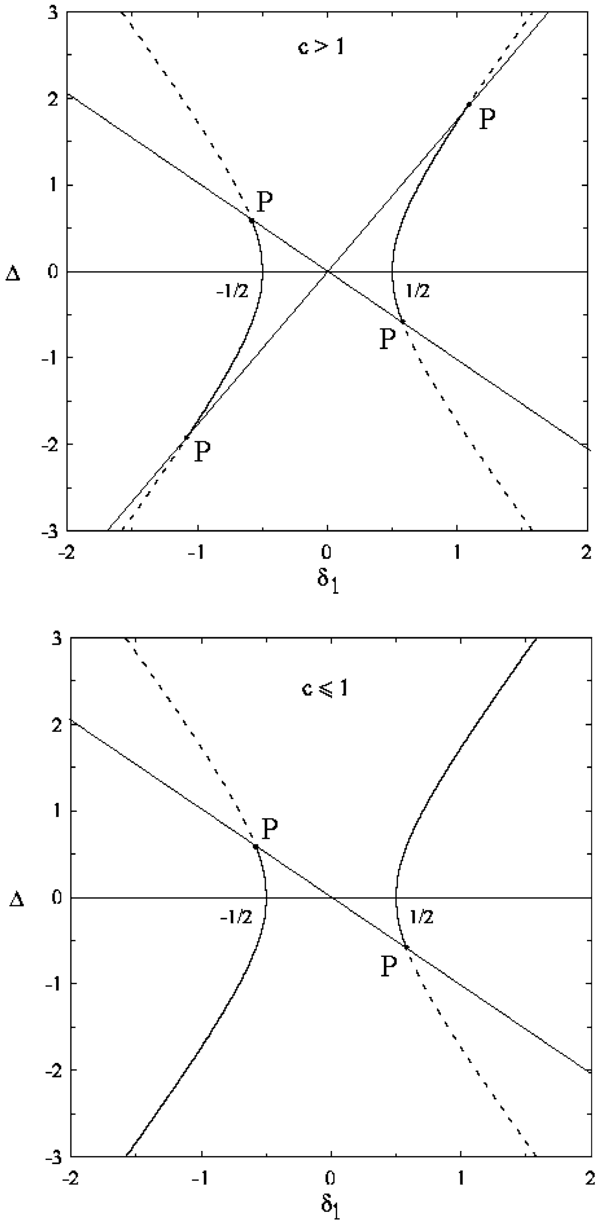

Figure 3. The resonance curves (18) and the straight lines (27) plotted in the δ_1–Δ plane for $c > 1$ (top) and $c \le 1$ (bottom). The intersection points P mark the location of pitchfork bifurcations. Quasiperiodic motions (s.s. flow equilibria) exist only on those portions of the resonance curves that are drawn solid. The origin is unstable on the solid portions of the resonance curves, and stable on the dashed portions.

We may further examine the effect of damping c on the existence of s.s. flow equilibria by solving $\sin 2\theta = \pm 1$ in Equation (26) for Δ:

$$\Delta = \frac{s}{2\sqrt{c}\sqrt{c-1}} \quad \text{and} \quad \Delta = -\frac{s}{2\sqrt{c}\sqrt{c+1}} \tag{30}$$

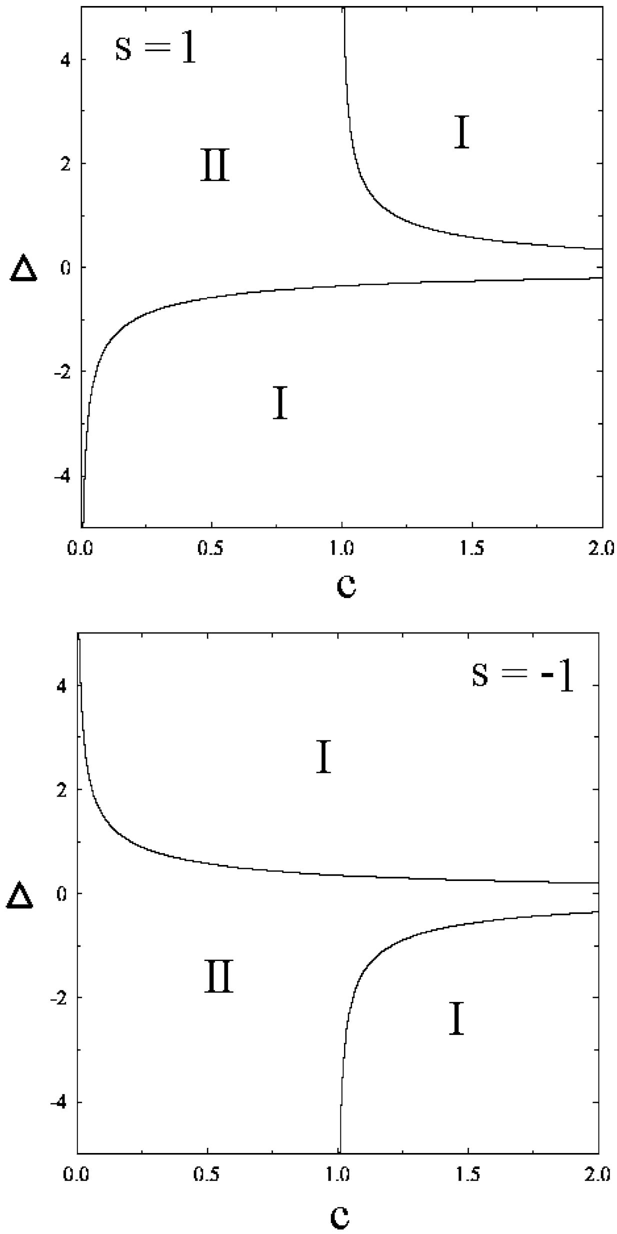

Figure 4. Bifurcation curves in the c–Δ plane of s.s. (slow slow) flow (21)–(22) for $s = 1$ (top) and $s = -1$ (bottom). Here, as in Equation (17), $s = 1$ corresponds to the right branch of the dashed curves in Figure 2, while $s = -1$ corresponds to the left branch. In the region I, in both figures, there exists only a stable trivial solution. In region II, the trivial solution becomes unstable and coexists with two nontrivial stable fixed points. A pitchfork bifurcation occurs by crossing region I to region II.

Equations in (30) plot as curves in the c–Δ parameter plane, see Figure 4. For a given value of c, the plots in Figure 4 give the values of Δ, or equivalently, of $\omega = 1 + \varepsilon\Delta$, for which the nontrivial pair of s.s. flow equilibria exist. Here again we see that when the damping coefficient $c > 1$, only a finite portion of the resonance curve (17) corresponds to nontrivial slow flow equilibria, whereas when $c \leq 1$, the corresponding portion of the resonance curve is infinite. Cf. Figures 3 and 4.

Now solving the RHS of Equation (25) for the amplitude r, and considering Equation (26), we obtain:

$$r = \frac{\sqrt{2}|\Delta|\,[(\Delta - 2c\Delta + 2\delta_1)(\Delta + 2c\Delta + 2\delta_1)]^{1/4}}{\left[3(2\Delta^2 + 3)|\gamma(2\delta_1 + 1)|\right]^{1/2}} \tag{31}$$

where δ_1 is given by Equation (17). For a real solution r in Equation (31) we must have

$$(\Delta - 2c\Delta + 2\delta_1)(\Delta + 2c\Delta + 2\delta_1) > 0 \tag{32}$$

Note that this inequality is related to Equation (27), i.e., the nontrivial s.s. flow equilibria must lie in region II of Figure 4.

In Figure 5, we show the amplitude r as given by Equation (31) as a function of the detuning Δ for both cases $c \leq 1$ and $c > 1$ and for left and right branches of the resonance curve (17), $s = -1$ and $s = 1$, respectively. It is interesting to note that although the two branches of the resonance curve (17) are point symmetric in the parameter plane with respect to the point ($\delta = 0.25,\ \omega = 1$) (Figure 2), that is, to ($\delta_1 = 0,\ \Delta = 0$), the nonlinear behavior along these curves does not share this symmetry. This may be seen directly from Equations (11) and (12) by noticing that these equations are invariant under the transforation

$$\Delta \mapsto -\Delta, \quad \delta_1 \mapsto -\delta_1, \quad A \mapsto B, \quad B \mapsto A, \quad \gamma \mapsto -\gamma \tag{33}$$

Thus, the left ($s = -1$) branch of the resonance curve (17) in a system with parameter γ will exhibit the same dynamical behavior as the right branch ($s = 1$) would using parameter $-\gamma$. Only in the case of $\gamma = 0$ (no nonlinear terms in Equation (5)) will the left and right branches have the same behavior dynamical behavior.

3. Conclusions

In this paper, we have studied the existence and bifurcation of QP solutions in the vicinity of the principal resonance 2:2:1 in the damped nonlinear QP Mathieu equation. We have used the two-variable expansion perturbation method, in the small ε limit, and derived the first slow flow parametric system. In a second step, we have performed a second perturbation expansion, for small μ, on the slow flow to investigate QP solutions in the neighborhood of the resonant curve lying in the middle of the stability region of the linear QP Mathieu equation (see Figure 2). The equilibria of the resulting s.s. (slow slow) flow autonomous system corresponding to quasiperiodic motion of the original equation were studied and the effect of damping on the existence of these QP motions near the resonant curve was discussed.

Our investigation focused on the middle of the principal instability region associated with 2:2:1 resonance [3], as shown in Figure 2. Figure 4 shows that the increase of the damping in Equation (5) restricts the zone of instability of the origin as seen in Figure 3. A pitchfork bifurcation in s.s. flow equilibria occurs as we move from region I to region II in Figure 4. The stable trivial solution, which is the only steady state in region I in Figure 4, becomes unstable and two stable spirals are born as we enter region II.

We note that the conditions (25) for the existence of quasiperiodic motions do not depend upon the nonlinearity coefficient γ. This may be explained by the fact that the pair of s.s. flow equilibria corresponding to these quasiperiodic motions only exist when the s.s. flow equilibrium at the origin is unstable. But the stability of the origin depends only on local conditions near the origin and thus

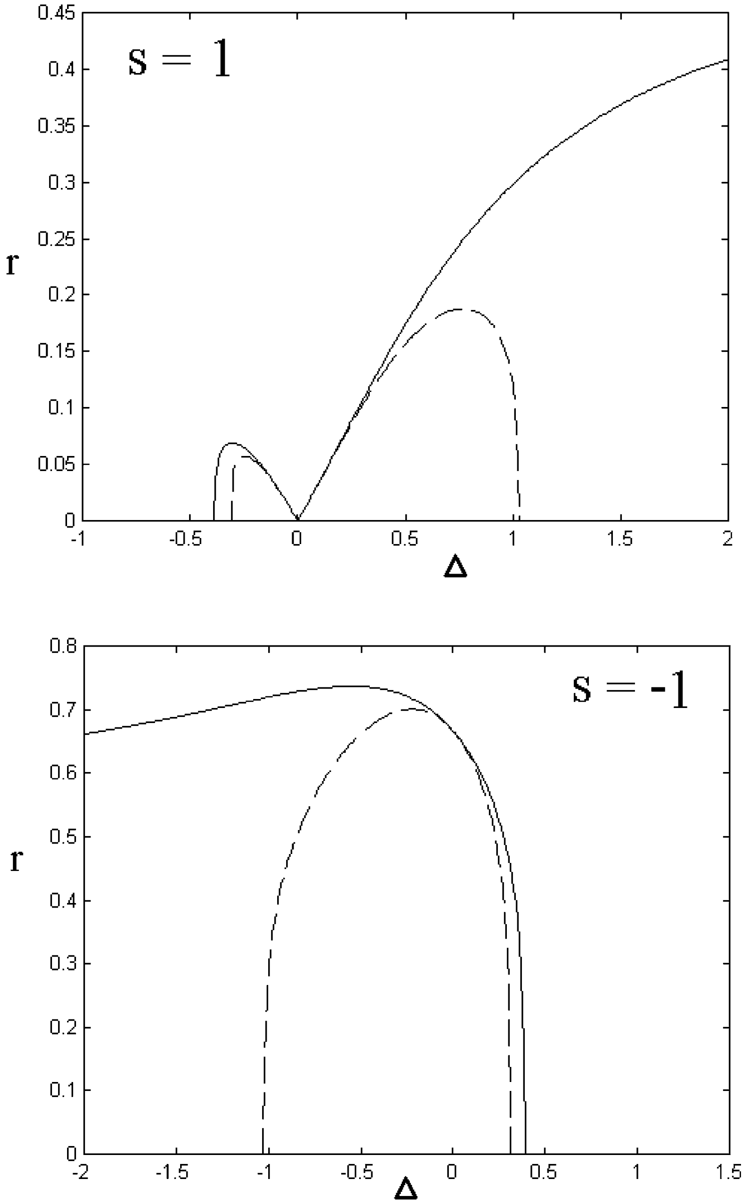

Figure 5. The amplitude r of the quasiperiodic motion, as given by Equation (31), plotted as function of the detuning parameter Δ. Solid line is $c = 0.9$, dashed line is $c = 1.2$. Here $\gamma = 1$.

does not depend on the nonlinearity coefficient. On the other hand, the amplitude r of the associated quasiperiodic motion does depend on γ, see Equation (31).

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