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ON THE EXISTENCE AND BIFURCATION OF MINIMAL NORMAL MODES

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Abstract—Minimal normal modes (MNMs) are defined as non-linear normal modes which give a true minimum to Jacobi's Principle of Least Action. It is shown that for a certain class of two degree of freedom non-linear conservative systems, MNMs generically occur in pairs. The nature of both generic and non-generic bifurcations of MNMs is derived and illustrative examples are given.

INTRODUCTION

In this paper we shall consider a class of non-linear, holonomic, scleronomous, conservative two degree of freedom systems with the following properties:

Let x, y be generalized coordinates for a system S. Then we take the kinetic energy T in the form

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{1}$$

where dots represent differentiation with respect to time, t.

The potential energy $V = V(x, y)$ is assumed to satisfy the following conditions [1, pp. 160, 179]:

(a) $V(0,0) = 0$,
(b) $V(-x, -y) = V(x, y)$,
(c) the partial derivatives V_x and V_y simultaneously vanish only at the origin, $V_x(0,0) = V_y(0,0) = 0$,
(d) $V(x, y)$ is positive definite,
(e) the curves $V(x, y) =$ constant form a continuum of smooth, simple, closed, non-intersecting curves surrounding the origin.

In particular, these assumptions imply that the origin 0 is a stable equilibrium point. Any trajectory of S which passes through 0 must, from condition (b), be symmetric with respect to 0 in the xy plane.

The motion of the system S is governed by the differential equations

$$\ddot{x} = -V_x, \tag{2}$$

$$\ddot{y} = -V_y. \tag{3}$$

These equations possess the first integral

$$T + V = h \tag{4}$$

where h is a constant equal to the total energy. Since $T \geq 0$, the motion must remain in the region of the xy plane which satisfies

$$V(x, y) \leq h. \tag{5}$$

For such a system S, non-linear normal modes have been defined by Rosenberg [1, pp. 162, 173]. In a manner similar to Rosenberg we take non-linear normal modes to be periodic motions which pass through the origin 0 and which have two rest points (located symmetrically with respect to 0) on the curve $V(x, y) = h$. See Fig. 1. (In the notation of [14] these periodic motions have been called BOB solutions.)

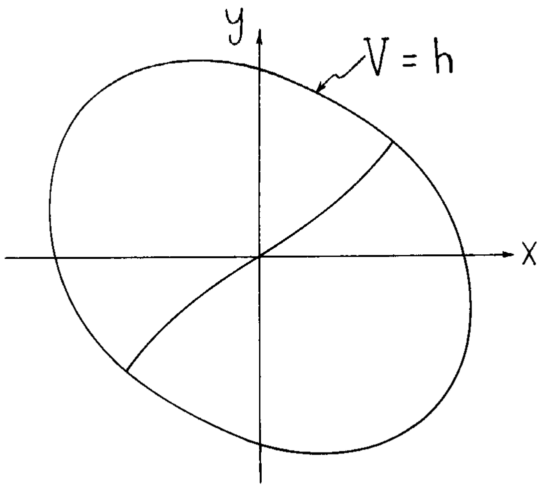

Fig. 1. A non-linear normal mode.

This paper considers a special class of non-linear normal modes called minimal normal modes which we define as follows:

Definition

A non-linear normal mode will be called a minimal normal mode (MNM) if it gives a true local minimum to Jacobi's Principle of Least Action. (Note: Jacobi's Principle of Least Action will be reviewed below.)

This class of periodic motions (MNMs) has emerged as being interesting because of the availability of results concerning their properties. Not long ago the existence and location of MNMs was implicitly considered [12], [2], but no specific definition was given. The results in [2] have been recently extended in [3].

This paper presents some new results concerning generic existence and bifurcation of MNMs. In what follows we will use the word 'generic' to mean typical or usual; for a precise topological definition see [6].

MINIMALITY OF JACOBI'S PRINCIPLE

Jacobi's Principle of Least Action [11], [13] states that among all paths in the xy plane between two fixed endpoints, A and B, a path actually followed by the system S, for fixed energy h, will give the action I a *stationary* value compared to all neighboring paths through A and B. Here the action I is defined as

$$I = \int_A^B \sqrt{h-V}\,\mathrm{d}s \tag{6}$$

where $\mathrm{d}s^2 = 2T\mathrm{d}t^2 = \mathrm{d}x^2+\mathrm{d}y^2$ from equation (1). Note that equation (6) makes no reference to the system's motion time t, but rather is based upon the geometry of the path in the xy plane.

As is well known from the calculus of variations, the stationary requirement is satisfied by solutions to the Euler-Lagrange equation associated with the functional I. For this problem, the Euler-Lagrange equation is well-known to take the form [1], [15]

$$2(h-V)y''+(1+y'^2)(V_y-y'V_x) = 0 \tag{7}$$

where $y = y(x)$ and where primes represent differentiation with respect to x.

Now suppose we have a solution to equation (7). Under what conditions will this path represent a true minimum for I (as opposed to only a stationary value) relative to all neighboring paths through the fixed endpoints? This question was first answered by Jacobi. We summarize his results but omit the derivation; see [4], [5] for details. Note that we consider only weak variations.

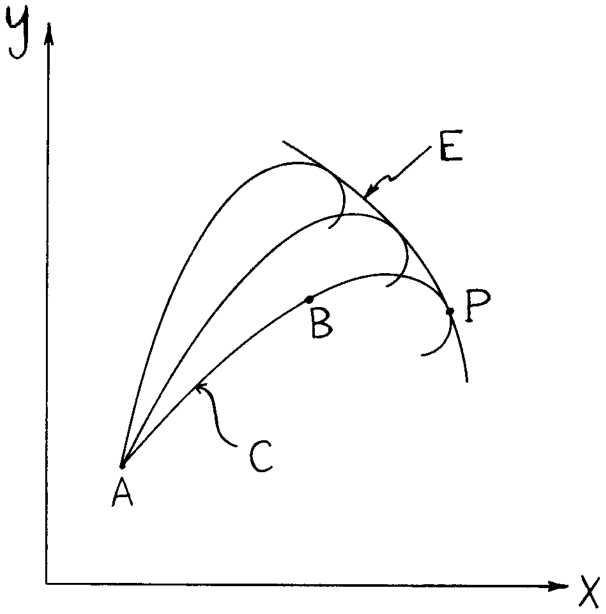

Fig. 2. Jacobi's Principle gives a true minimum if the second endpoint B lies between A and P on C. See text.

Let E be the first envelope of a family of paths which are solutions to equation (7) and which all pass through point A. Let a given path C in this family be tangent to E at a point P. (See Fig. 2.)

If the second endpoint B of equation (6) lies between A and P on C, then C furnishes I with a true minimum. If, on the other hand, B lies at or beyond P on C, and if E is smooth at P, then there exist neighboring paths through A and B which give I a smaller value than C does. (If E has a cusp at P, however, the curve C may still give I a true minimum, even if point B lies beyond P; see [4, p. 59, footnote].)

In order to apply these results to MNMs, we take $A = 0$, the origin. Now consider the first envelope E of all trajectories of S emanating from 0 for fixed energy h. From condition (b) above, E (like the trajectories themselves) will be symmetric with respect to 0.

Now suppose that E is tangent to the $V = h$ curve at the point Q. Then E will also be tangent to $V = h$ at a point Q', symmetric to Q with respect to 0.

In such a case, $Q0Q'$ will be a MNM in the following sense: if point B of equation (6) is identified with a point on $Q0Q'$ arbitrarily close to Q, then $0B$ will give Jacobi's Principle a true minimum with respect to all neighboring paths through the fixed endpoints 0, B. In order to obtain a true minimum, point Q, which lies on E, must be omitted.

Note, however, that I evaluated between the endpoints 0 and B gives the same value as I between 0 and Q.

A LEMMA

In this section we derive a basic lemma which will be used to discuss generic behavior of MNMs.

Let α be the angle which a trajectory C emanating from the origin 0 meets the x axis at 0. We shall exhibit a function $f(\alpha)$ which vanishes precisely when α corresponds to a MNM.

To begin with, consider the family of trajectories which emanate from 0 at a fixed energy h. Let α parametrize this family. Due to the symmetry assumed in condition (b) above, we may restrict our attention to $0 \leqslant \alpha < \pi$.

Now let G be any smooth closed curve which surrounds 0 and which is symmetric with respect to 0. (See Fig. 3.)

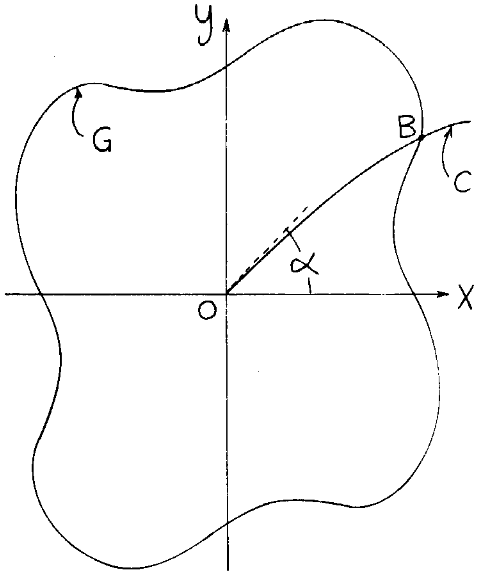

Fig. 3. G is any smooth closed curve which surrounds 0 and which is symmetric with respect to 0. C is the trajectory which emanates from 0 at energy h, making an angle α with the x axis.

Let

$$I(\alpha) = \int_0^B \sqrt{h-V}\, \mathrm{d}s \tag{8}$$

where the integral is evaluated along a trajectory C which emanates from 0 at angle α. Here B is the point of intersection of C with G. (We assume that G is chosen such that B exists for every α.)

Now consider the value of $I(\alpha')$ along a trajectory C' which neighbors C, i.e. for α' nearly equal to α. In Appendix I the following expression is derived relating $I(\alpha')$ and $I(\alpha)$:

$$I(\alpha') = I(\alpha) + \varepsilon\sqrt{h-V_B}\, \hat{T}_B \cdot \bar{\eta}_B + 0(\varepsilon^2) \tag{9}$$

where

V_B = *V evaluated at point B*,
$\hat{T}_B$ = unit tangent vector to C at B,
$\varepsilon\bar{\eta}_B$ = vector from B to B' (where B' is the point of intersection of C' with G).

As α' approaches α, $\varepsilon\bar{\eta}_B$ approaches $\hat{\tau}_B \mathrm{d}l$, where $\hat{\tau}_B$ is a unit tangent vector to G and B and where $\mathrm{d}l$ is the element of arc along G. Equation (9) then becomes

$$\frac{\mathrm{d}I(\alpha)}{\mathrm{d}\alpha} = \frac{\mathrm{d}l}{\mathrm{d}\alpha}\sqrt{h-V_B}\cos\theta \tag{10}$$

where $\cos\theta = \hat{T}_B \cdot \hat{\tau}_B$, i.e. θ is the angle between C and G at B.

For convenience in what follows, let

$$f(\alpha) = \frac{\mathrm{d}I(\alpha)}{\mathrm{d}\alpha}. \tag{11}$$

Notice that $f(\alpha)$ is a periodic function of α with period π. From equation (10), $f(\alpha)$ must vanish when any one of the following three conditions is met:

(i) $\mathrm{d}l/\mathrm{d}\alpha = 0$,
(ii) $\cos\theta = 0$,
(iii) $V_B = h$.

As demonstrated in Appendix II, condition (i) implies that the curve G typically has a cusp at B and hence condition (i) cannot generically be fulfilled if G is smooth. In what follows we assume $\mathrm{d}l/\mathrm{d}\alpha$ does not vanish on G.

Now choose $G = E$, the first envelope of trajectories emanating from 0. At most points on E, the envelope is tangent to a trajectory and $\hat{T}_B = \pm\hat{\tau}_B$. This gives $\cos\theta = \pm 1$ and condition (ii) is not satisfied.

At points on E where condition (ii) is satisfied, condition (iii) must also be satisfied. Otherwise a trajectory would pass orthogonally through its own envelope with non-zero velocity, which is impossible.

Thus the only way in which $f(\alpha)$ can vanish is if condition (iii) is satisfied. In this case, point B lies on the $V = h$ curve and the corresponding trajectory C is a MNM.

Moreover, every MNM will be included in the zeros of $f(\alpha)$. For although there may exist non-linear normal modes which originate at 0, touch E at a point away from the $V = h$ curve, and then continue on to eventually reach the $V = h$ curve, such trajectories will not give Jacobi's Principle a true minimum and hence will not be MNMs.

We therefore have the following:

Lemma

Let S be a system as above with a smooth closed first envelope E of trajectories emanating from 0 for fixed energy h. Then a necessary and sufficient condition that α correspond to a MNM is that

$$f(\alpha) = \mathrm{d}I(\alpha)/\mathrm{d}\alpha = 0.$$

GENERIC BEHAVIOR OF MNMs

By this lemma we have a one to one correspondence between MNMs and the zeros of a periodic function $f(\alpha)$. The question of the generic existence of MNMs is therefore reduced to the question: how may the zeros of $f(\alpha)$ typically be distributed on $0 \leqslant \alpha < \pi$? Similarly if we vary some system parameter (including h, for example) and thereby change the function $f(\alpha)$, then the question of generic bifurcation of MNMs becomes: how may new zeros of $f(\alpha)$ be typically introduced?

We shall approach genericity via the notion of transversality. The curve $f = f(\alpha)$ will be said to intersect the line $f = 0$ transversally if $f'(\alpha) \neq 0$ at the point of intersection. (See Fig. 4.)

Transversality is an important concept in differential topology [6], [7]. In a word, transversality is generic. Although this statement can be proved rigorously via Thom's Transversality Theorem [6], the plausibility of the genericity of transversality may be established as follows: for a non-transversal intersection, both $f(\alpha)$ and $f'(\alpha)$ must simultaneously vanish. Two algebraic equations in one unknown 'typically' have no solution.

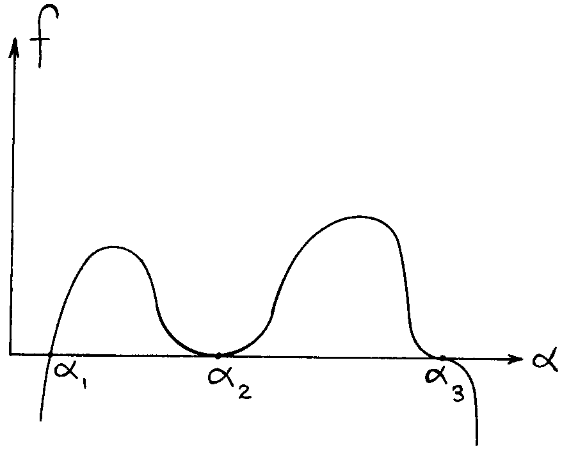

Fig. 4. The curve $f = f(\alpha)$ intersects the line $f = 0$ transversally at α_1 but not at α_2 or α_3.

In order to apply these ideas to MNMs, consider the function $f(\alpha)$ on $0 \leqslant \alpha < \pi$. If all intersections of $f = f(\alpha)$ with $f = 0$ occur transversally (the generic case), then as $f(\alpha)$ has period π, the total number of zeros of $f(\alpha)$ must be even.

Theorem 1

(Existence): MNMs generically occur in pairs.

Note that this and the following theorems are only applicable to systems S with smooth closed first envelopes E about 0 as in the lemma.

Theorem 1 was largely anticipated by Yen [8] who claimed however that non-linear normal modes *always* occur in pairs. We present an example (non-generic) with three MNMs in the next section.

In order to consider bifurcations of MNMs we must place ourselves in the non-generic situation in which $f(\alpha)$ and $f'(\alpha)$ simultaneously vanish (points α_2 and α_3 of Fig. 4, for example). This is because in the transversal case, a small change in $f(\alpha)$ will not change the number of roots of $f(\alpha) = 0$.

The most typical non-generic case will involve

$$f(\alpha) = 0, \quad f'(\alpha) = 0, \quad \text{but} \quad f''(\alpha) \neq 0 \tag{12}$$

which corresponds to point α_2 of Fig. 4. If $f(\alpha)$ is changed slightly then 0, 1 or 2 MNMs may exist in the vicinity of the double root α of equation (12). (See Fig. 5.)

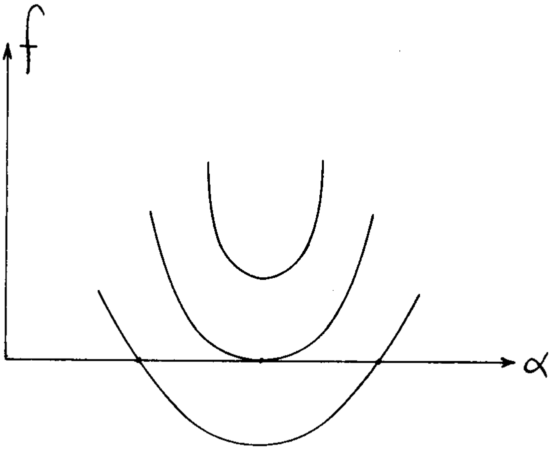

Fig. 5. A slight change in $f(\alpha)$ can produce 0, 1 or 2 MNMs in the vicinity of the double root α of equation (12).

Theorem 2

(Generic bifurcation): The generic bifurcation of a MNM may be described as follows: a MNM appears "out of nowhere" and splits into two MNMs.

An example illustrating this and the following theorem will be given in the next section.

In order to consider the next most likely case, suppose that at some point α,

$$f(\alpha) = 0, \quad f'(\alpha) = 0, \quad f''(\alpha) = 0, \quad \text{but} \quad f'''(\alpha) \neq 0. \tag{13}$$

This corresponds to point α_3 of Fig. 4. Now a small change in $f(\alpha)$ can produce 1 or 3 MNMs in the vicinity of the triple root α of equation (13). (See Fig. 6.)

Theorem 3

(Non-generic bifurcation): The most typical non-generic bifurcation of a MNM may be described as follows: an existing MNM throws off two new MNMs and continues to exist itself.

This type of bifurcation has been reported by several researchers [8], [9], [10]. The systems in which this nongeneric bifurcation occurs often possess some special symmetry.

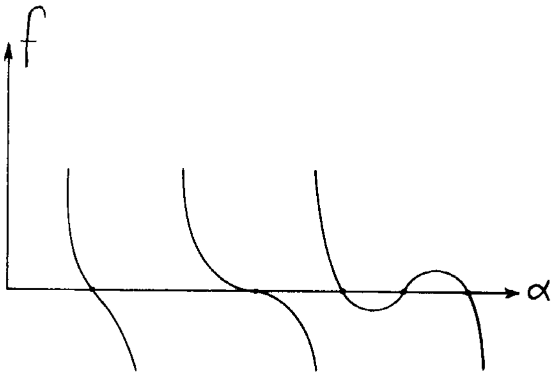

Fig. 6. A slight change in $f(\alpha)$ can produce 1 or 3 MNMs in the vicinity of the triple root α of equation (13).

For example, in the case of two unit masses constrained to a straight line and restrained by three springs, the anchor springs may be identical.

Even more complicated types of bifurcations may be generated in a hierarchical fashion by requiring $f(\alpha)$ and its first n derivatives to simultaneously vanish. These become progressively less generic as n increases and shall not be considered here.

EXAMPLES

Consider a system S for which

$$V(x,y) = \tfrac{1}{2}(a\xi^2 + b\eta^2) + \tfrac{1}{4}(3x^4 + y^4) \tag{14}$$

where

$$\begin{aligned} \xi &= \xi(x,y) = x\cos\theta + y\sin\theta \\ \eta &= \eta(x,y) = -x\sin\theta + y\cos\theta. \end{aligned} \tag{15}$$

Here a, b, θ are parameters. The ξ, η axes are rotated by an angle θ with respect to the x, y axes.

The trajectories of this system were obtained by numerical integration of equations (2), (3). In particular, the family of trajectories emanating from 0 at fixed energy h was studied by varying the angle α that a trajectory made with the x axis at 0.

Case 1

An example of the generic bifurcation of Theorem 2. Take $a = 8$, $b = 4$, $\theta = 45°$.

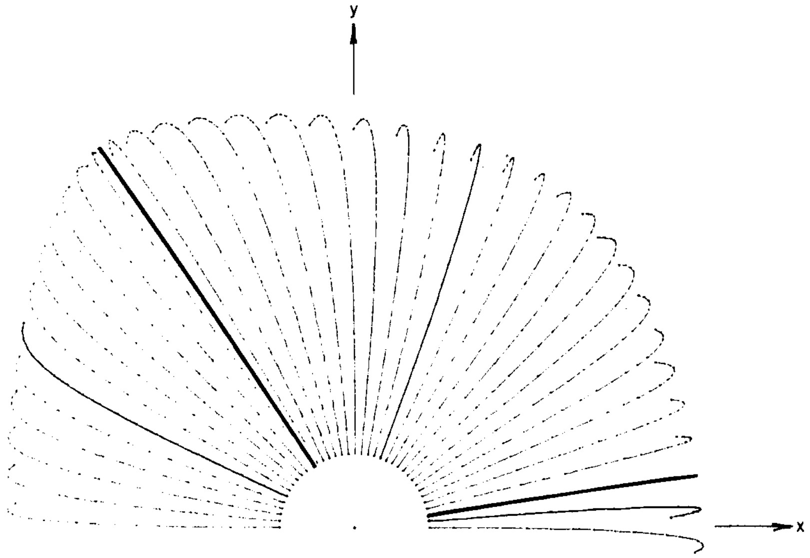

Fig. 7. Trajectories which originate at 0 for Case 1, $a = 8$, $b = 4$, $\theta = 45$, $h = 40$.

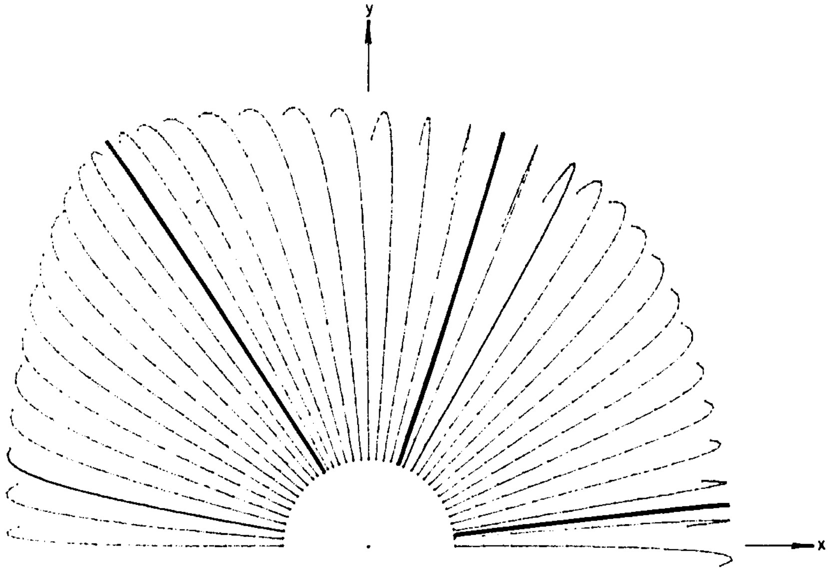

Fig. 8. Trajectories which originate at 0 for Case 1, $a = 8$, $b = 4$, $\theta = 45$, $h = 70$. Note the bifurcation at $\alpha = 71$.

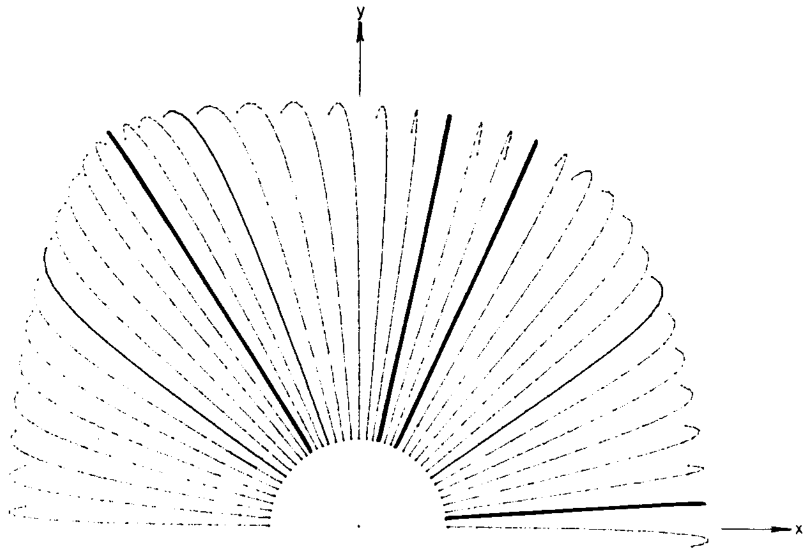

Fig. 9. Trajectories which originate at 0 for Case 1, $a = 8$, $b = 4$, $\theta = 45$, $h = 100$. The MNMs at $\alpha = 65°, 77°$ have bifurcated out of the MNM at $\alpha = 71°$ in Fig. 8.

Figures 7, 8 and 9, correspond to energies $h = 40$, 70 and 100, respectively. Note that portions of the trajectories near the origin 0 have not been plotted for convenience. Due to symmetry only $0 \leqslant \alpha < \pi$ need be displayed.

Results concerning MNMs may be summarized as follows: for $h = 40$ (Fig. 7) there are 2 MNMs at $\alpha = 10°$ and $123°$. As h is increased, there appears at about $h = 70$ (Fig. 8) a new MNM at $\alpha = 71°$. Further increases in h cause this MNM to split into two. For example, at $h = 100$ (Fig. 9), we find 4 MNMs at $\alpha = 5°, 65°, 77°$ and $122°$.

Case 2

An example of the non-generic bifurcation of Theorem 3. Again, take $a = 8$ and $b = 4$, but now choose $\theta = 0°$. Here the ξ, η axes and the x, y axes coincide and the system possesses a special degree of symmetry.

In fact this system possesses two similar (straight line) normal modes at $\alpha = 0, 90^\circ$ for every $h > 0$. For values of h which are small enough, these are found to be the only MNMs (Fig. 10, $h = 10$). As h is increased, two new MNMs bifurcate out of the one at $\alpha = 90^\circ$. Figure 11 at $h = 24$ shows the situation after bifurcation; there are MNMs at $\alpha = 0^\circ, 76^\circ, 90^\circ, 104^\circ$.

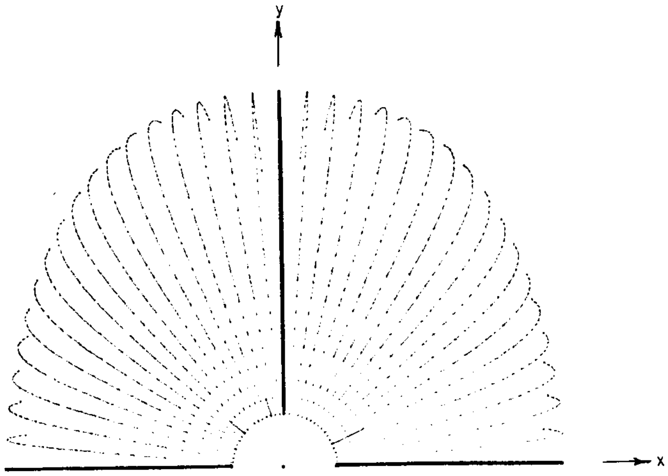

Fig. 10. Trajectories which originate at 0 for Case 2, $a = 8$, $b = 4$, $\theta = 0^\circ$, $h = 10$.

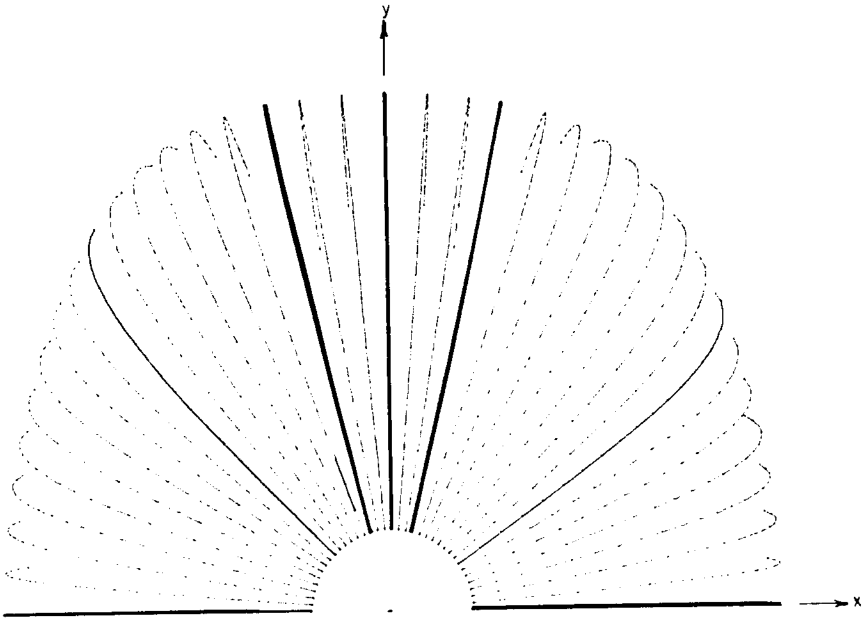

Fig. 11. Trajectories which originate at 0 for Case 2, $a = 8$, $b = 4$, $\theta = 0$, $h = 24$. Note MNMs at $\alpha = 0^\circ, 76^\circ, 90^\circ, 104^\circ$.

Case 3

The purpose of this example is to display a system for which the lemma and theorems of this paper are inapplicable. Take $a = 32$, $b = 2$, $\theta = 45^\circ$ and $h = 10$ (see Fig. 12).

Here the first envelope E is cusped. It can no longer be concluded than MNMs occur in pairs, and in fact there is only one MNM at $\alpha = 44^\circ$.

Note the other periodic motions at $\alpha = 123^\circ$ and 190°. These are not MNMs since each touches the envelope E before it reaches the $V = h$ curve.

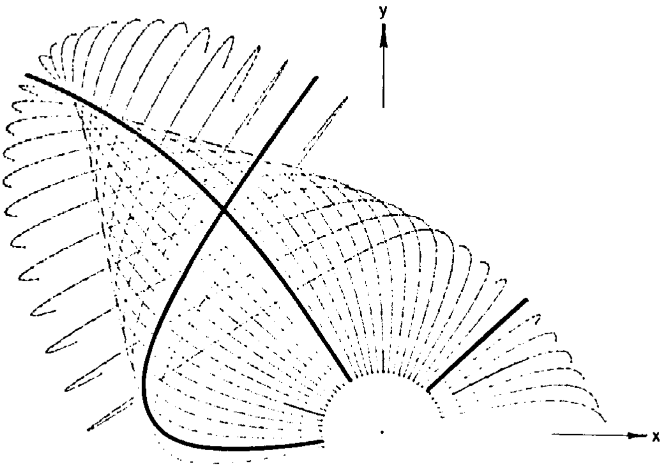

Fig. 12. Trajectories which originate at 0 for Case 3, $a = 32$, $b = 2$, $\theta = 45°$, $h = 10$. Note MNM at $\alpha = 44°$. The motions at $\alpha = 123°$, $190°$ are periodic but are not MNMs.

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APPENDIX I

Let C be a trajectory reaching from 0 to B and let C' be a neighboring trajectory reaching from 0 to B'. Let I on C be given by equation (8), while

$$I' = \int_0^{B'} \Phi' \, ds' \tag{A1}$$

where $\Phi = (h - V)^{1/2}$ and where primes denote values taken on C'.

Now let s be the distance along trajectory C from 0. The position vector $\bar{r}'$ from 0 to a point on C' may be related to the corresponding vector $\bar{r}$ of C by the equation

$$\bar{r}'(s) = \bar{r}(s) + \varepsilon\bar{\eta}(s). \tag{A2}$$

At point B, $\varepsilon\bar{\eta}$ reaches from B to B'. At the origin 0, $\bar{\eta}$ must vanish since both C and C' pass through 0,

$$\bar{\eta}(0) = 0. \tag{A3}$$

From equation (A2),

$$\left(\frac{ds'}{ds}\right)^2 = \frac{d\bar{r}'}{ds}\cdot\frac{d\bar{r}'}{ds} = 1 + 2\varepsilon\hat{T}\cdot\frac{d\bar{\eta}}{ds} + 0(\varepsilon^2) \tag{A4}$$

where

$$\hat{T} = \frac{d\bar{r}}{ds} = \text{unit tangent to } C.$$

Therefore

$$ds' = \left[1 + \varepsilon\hat{T}\cdot\frac{d\bar{\eta}}{ds} + 0(\varepsilon^2)\right]ds. \tag{A5}$$

Now develop Φ' in a Taylor series about $\varepsilon = 0$,

$$\Phi' = \Phi + \nabla\Phi\cdot\varepsilon\bar{\eta} + 0(\varepsilon^2). \tag{A6}$$

Substituting equations (A5) and (A6) into (A1), find

$$I' = I + \varepsilon\int_0^B\left[\nabla\Phi\cdot\bar{\eta} + \Phi\hat{T}\cdot\frac{d\bar{\eta}}{ds}\right]ds + 0(\varepsilon^2). \tag{A7}$$

Integrating by parts,

$$I' = I + \varepsilon\int_0^B\left[\nabla\Phi - \frac{d}{ds}(\Phi\hat{T})\right]\cdot\bar{\eta}\,ds + \varepsilon\,\Phi\hat{T}\cdot\bar{\eta}\Big|_0^B + 0(\varepsilon^2). \tag{A8}$$

Since C, C' are trajectories, the integrand vanishes by Jacobi's Principle. The equation

$$\nabla\Phi - \frac{d}{ds}(\Phi\hat{T}) = 0 \tag{A9}$$

can be shown to be equivalent to equation (7).

Using equations (A3), (A9) in (A8) we obtain equation (9) above.

APPENDIX II

Let G be a curve in the xy plane parametrized by α, and let $dl^2 = dx^2 + dy^2$ be the line element along G. We show that if

$$\frac{dl}{d\alpha} = 0 \tag{B1}$$

at some point B on G, then G typically has a cusp at B.

Let $\bar{r} = \bar{r}(\alpha)$ be a position vector from 0 to a point of G. Then

$$\left(\frac{dl}{d\alpha}\right)^2 = \frac{d\bar{r}}{d\alpha}\cdot\frac{d\bar{r}}{d\alpha} \tag{B2}$$

and equations (B1), (B2) give at B

$$\frac{d\bar{r}}{d\alpha} = 0. \tag{B3}$$

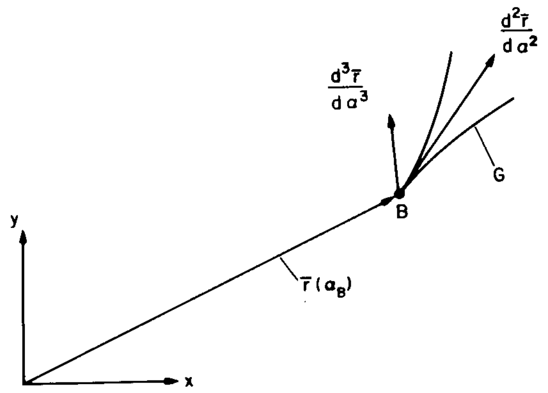

Fig. 13. If $d\mathbf{r}/d = 0$ and if $d^2\mathbf{r}/d\alpha^2$, $d^3\mathbf{r}/d\alpha^3$ are linearly independent at B, then G has a cusp at B from equation (B4).

Expanding $\bar{\mathbf{r}}(\alpha)$ in a Taylor series about $B(\alpha = \alpha_B)$,

$$\bar{\mathbf{r}}(\alpha) = \bar{\mathbf{r}}(\alpha_B) + \frac{1}{2}\frac{d^2\bar{\mathbf{r}}}{d\alpha^2}\bigg|_B (\Delta\alpha)^2 + \frac{1}{6}\frac{d^3\bar{\mathbf{r}}}{d\alpha^3}\bigg|_B (\Delta\alpha)^3 + 0[(\Delta\alpha)^4] \tag{B4}$$

where $\Delta\alpha = \alpha - \alpha_B$.

Now B has already been characterized by $d\bar{\mathbf{r}}/d\alpha = 0$, and therefore no additional arbitrary (but fixed) relationship will generically be satisfied by $\bar{\mathbf{r}}(\alpha)$. In particular,

$$\frac{d^2\bar{\mathbf{r}}}{d\alpha^2} \quad \text{and} \quad \frac{d^3\bar{\mathbf{r}}}{d\alpha^3}$$

will typically be linearly independent vectors at B.

In this case, equation (B4) reveals that G has a cusp at B. (See Fig. 13.)

Résumé:

On définit les modes normaux minimaux (MNM) comme des modes normaux non linéaires qui donnent un vrai minimum au principe de moindre action de Jacobi. On montre que pour un certain ensemble de systèmes conservatifs non linéaires à deux degrés de liberté, les MNM apparaissent alors, de façon caractéristique, par paires. On établit la nature des bifurcations caractéristiques ou non des MNM et on donne des exemples pour l'illutrer.

Zusammenfassung:

Minimale Normalschwingungsformem (MNM) werden als nichtlineare Normalschwingungsforem definert, die dem Prinzip der geringsten Wirkung nach Jacobi ein wahres Minimum geben. Es wird gezeigt, dass für eine bestimmte Klasse nichtlinearer konservativer System mit zwei Freiheitsgraden die MNM gattungsweise in Paaren auftreten. Das Wesen der gattungsgleichen und gattungsungleichen Aufspaltung der MNM wird bestimmt und illustrierende Beispiele werden angegeban.