Limb Impact Harvesting, Part I: Finite Element Analysis

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ABSTRACT

A finite element model of the dynamics of limb impact harvesting is presented. The model is based upon linearized beam theory and accounts for transverse shear and includes consideration of the leaves, twigs, secondary branches and fruits. The specific fruit of interest is modelled as a spherical pendulum. Newmark's direct integration scheme was used to obtain the *transient* response.

INTRODUCTION AND REVIEW OF LITERATURE

The desire to reduce mechanical damage to fruit has led to the concept of impact harvesting. The hypothesis is that it is possible for the base of the limb to be subjected to an impact, while the fruit remains relatively stationary with detachment resulting. We shall present an analysis of this behavior.

The specific objective of this paper is to develop a finite element model to study the transient dynamics of the fruit-stem subsystem when the base of the limb is impacted. Mathematical modelling of the fruit harvesting system is expected to provide a rational basis for the design of harvesting equipment and to also provide an adequate explanation to often confusing experimental results.

Fridley and Adrian (1960), Adrian and Fridley (1963), Fridley and Lorenzen (1965), Yung and Wang (1968), Phillips, Hutchinson and Fridley (1970), Hoag, Hutchinson and Fridley (1970), Schuler and Bruhn (1973), Hussain, Rehkugler and Gunkel (1975) and Ghate and Rohrbach (1975) have studied the energy transfer through the limb or trunk of different fruit crops under various loading conditions.

Several other researchers such as Thomas (1963), Wang (1965a, 1965b), Studer (1966), Wang and Shellenberger (1967), Rumsey (1967), Cooke and Rand (1969), Rand and Cooke (1970), Parchomchuk and Cooke (1972) and Stafford and Diener (1971, 1973) have studied the behavior of the fruit-stem subsystem when subjected to sinusoidal vibration. Abu-Gheida, Stout and Ries (1962) and Diener, Mohsenin and Jenks (1963) have also considered the presence of fruits in their models.

The development of the finite element method in recent years has made it technically feasible to investigate the combined system consisting of both the tree-limb and the fruit-stem subsystems. Yung and Fridley (1975) and Fridley and Yung (1975) investigated the steady state behavior of the entire tree system subjected to a sinusoidal vibration, using a finite element method. Although Yung and Fridley (1975) analyzed the steady state behavior, they recognized the importance of the *transient* response during which most of the fruit is removed. A comprehensive bibliography of pertinent literature is provided at the end of this series of three papers.

MATHEMATICAL MODELLING

In this study we use the finite element method to investigate the *transient* behavior during limb impact harvesting. In impact harvesting only the transient behavior is of interest. Our model accounts for the deflection due to transverse shear as well as the usual bending effects. The specific fruit of interest is modelled as a spherical pendulum with six degrees of freedom.

The natural geometry of the limb is accounted for in this study. To reduce the computational costs, the leaves, twigs, secondary branches and all other fruits are treated as being distributed along the length of the limb in proportion to the area of the cross section of the element. That part of the limb which extends beyond the point of attachment for the specific fruit of interest is lumped to the last element of the limb and represented with an equivalent mass and damping coefficient.

If we remove all the leaves, twigs and fruits from a limb and divide the limb into small sections, the resulting elements can be assumed to be straight and homogeneous (Fig. 1). When the limb is impacted, the limb, and therefore each of these elements, vibrates. Under the assumption of small deflections, the motion of each of these elements can be decomposed into four independent motions (Fig. 1). These motions are:

1 Transverse vibration, v - motion along x axis
2 Axial vibration, u - motion along y axis
3 Transverse vibration, w - motion along z axis
4 Axial rotation, θ - rotation about y axis

The local coordinate system used in this study is an inertial Cartesian frame of reference with the y axis located

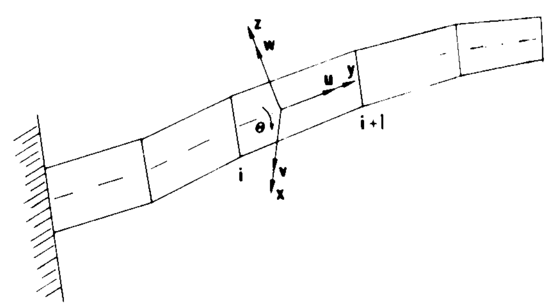

FIG. 1 A finite element discretization of a typical limb.

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along the beam axis of the element when the limb is at rest. The z axis is directed such that the yz plane is vertical. The local coordinate system is fixed in space and does not move with the element.

In order to apply the beam equations we make the following assumptions:

1 stress is proportional to strain
2 plane cross-sections remain plane
3 slope of the deflection curve is small compared to unity
4 rotatory inertia is negligible (Schuler and Bruhn, 1973)

Our analysis was based on small deflection theory. In those applications where large deflections are encountered, an analysis which includes mathematically more complicated nonlinear effects should be considered.

We can now apply either

1 Bernoulli-Euler beam theory, or
2 Timoshenko beam theory.

Timoshenko beam theory accounts for both the deflection due to transverse shear and the rotatory inertia. However, in this study we found that when Timoshenko beam theory was applied with two independent degrees of freedom ((i) total deflection and (ii) slope due to bending), the resulting matrix equation was *ill-conditioned* and the solution was *numerically unstable*. This resulted because the shear energy associated with the limb deflection was small.

Therefore, Bernoulli-Euler beam theory was used. This theory assumes a small deflection due to transverse shear. We have accounted for the deflection due to transverse shear by using a complementary energy approach in the development of the stiffness matrix (Gallagher, 1975). The application of Bernoulli-Euler theory to the limb vibration problem yielded the following four equations of motion for each element (Upadhyaya, 1979).

Motion (i) - transverse vibration, v

$$\rho A\ddot{v} + c_t\dot{v} + (EIv'')'' = F_x\delta(y-a) + M_x\delta'(y-a) + f_x\left\{\delta(y) + \delta(y-L_i)\right\} + m_x\left\{\delta'(y) + \delta'(y-L_i)\right\} \quad \ldots\ldots\ldots \quad [1]$$

Motion (ii) - axial vibration, u

$$\rho A\ddot{u} + c_a\dot{u} - (EAu')' = F_y\,\delta(y-a) + f_y\left\{\delta(y) + \delta(y-L_i)\right\} \quad \ldots\ldots\ldots\ldots \quad [2]$$

Motion (iii) - transverse vibration, w

$$\rho A\ddot{w} + c_t\dot{w} + (EIw'')'' = F_z\delta(y-a) + M_z\,\delta'(y-a) + f_z\left\{\delta(y) + \delta(y-L_i)\right\} + m_z\left\{\delta'(y) + \delta'(y-L_i)\right\} \quad \ldots\ldots \quad [3]$$

Motion (iv) - axial rotation, θ

$$\rho J\ddot{\theta} + c_r\dot{\theta} - (GJ\theta')' = T_y\,\delta(y-a) + t_y\left\{\delta(y) + \delta(y-L_i)\right\} \quad \ldots\ldots\ldots\ldots \quad [4]$$

where,

ρ = mass density, ML^{-3}
A = area of cross section, L^2
E = Young's modulus, $ML^{-1}T^{-2}$
I = area moment of inertia, L^4
L_i = length of the element, L
c_t, c_a, c_r = transverse, axial and rotational damping coefficients, MT^{-1}
G = shear modulus, $ML^{-1}T^{-2}$
J = polar moment of area, L^4
F_x, F_y, F_z = components of external force along the local coordinates, MLT^{-2}
M_x, M_z = components of external moment, ML^2T^{-2}
T_y = external torque, ML^2T^{-2}
f_x, f_y, f_z = components of the internal nodal forces, MLT^{-2}
m_x, m_z = components of internal nodal moments, ML^2T^{-2}
t_y = internal nodal torque, ML^2T^{-2}
$\delta(y-a)$ = Dirac delta function
a = location of the external force from the base of the element, L
$\cdot$ = partial derivative with respect to time
$'$ = partial derivative with respect to y.

The first term on the left hand side of equation [1] represents the inertia of the element, the second term represents damping and the third term represents the stiffness of the element. The first term on the right hand side represents the external impact load applied at a distance "a" from the base of the element. The second term is the derivative of the concentrated moment due to the eccentricity of impact. f_x and m_x are the internal nodal forces and moments arising from the interconnectivity of the elements.

Terms in equation [2] are quite similar to those in equation [1] except that the moment terms do not appear here.

Equation [3] is similar to equation [1].

Equation [4] is similar to equation [2] except that the area of the cross-section A (equation [2]) is replaced by the polar moment of area J (equation [4]).

The boundary conditions which must be satisfied are the assumed fixed-end conditions at the base of the limb (i.e., the displacements and the slopes in the local coordinate system are zero for transverse vibrations and likewise the displacements are zero for axial vibration and axial rotation) and the conditions of coupling of the fruit to the stem. The attachment of the fruit to the stem couples the six ordinary differential equations describing the motion of the fruit to the four partial differential equations describing the motion of the stem.

Consideration of Leaves, Twigs, Secondary Branches and Fruits

In this model we have assumed that the leaves, twigs, secondary branches and other fruits add to the inertia and that the leaves and twigs add to the damping. Although the external damping is reported to be proportional to $(\text{velocity})^{1.41}$ (Hoag, Fridley and Hutchinson, 1969), we have assumed linear damping to simplify the model. Phillips, Hutchinson and Fridley (1970) have found that randomly distributed leaves and twigs can be assumed to be uniformly distributed over the limb without introducing appreciable error. In our model a given element may have a linear taper in diameter in order to account for the natural taper of the limb. The proportional distribution of leaves, branches, twigs and fruit was accomplished by using effective densities and effective damping coefficients in equations [1] to [4].

Since the limb is only slightly tapered, this assumption is almost the same as the assumption of uniform distribution.

Equations of Motion for the Fruit-Stem Subsystem

The fruit stem was considered to be a series of elastic beam elements. The fruit of interest was coupled to the stem by torsional springs. This coupling gave three rotational and three translational degrees of freedom to the fruit. By continuity, the translational degrees of freedom are the same as that of the end of the stem. The rotational degrees of freedom correspond to precession, nutation and spin of the fruit. Although one could use an Eulerian angle approach (Greenwood, 1965, p. 333-335) to develop the equations of motion of the fruit, such an approach leads to non-linear equations of motion because the precession angle cannot be assumed to be small. Under the small angle assumptions, we can assume that the nutation angle (Γ) is the vector addition of α and β (Fig. 2). α corresponds to the rotation about the local z axis and β corresponds to the rotation about the x axis. The precession angle becomes $\tan^{-1}(\alpha/\beta)$. If ψ corresponds to the spin of the fruit, then α, β and ψ can be taken as an equivalent set of degrees of freedom under the small angle assumption. We further assume that

(i) the fruit is spherical in shape
(ii) the fruit is attached to the stem at a distance r from the center of gravity of the fruit.

The equations of motion can be obtained by applying Lagrange's equations (Greenwood, 1965, p. 252-258):

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}_i}\right) + \frac{\partial P}{\partial \dot{q}_i} - \frac{\partial \tilde{L}}{\partial q_i} = Q_i \quad \text{[5]}$$

where,

$\tilde{L}$ = Lagrangian
P = Raleigh dissipation function
Q_i = generalized force vector
q_i = i^{th} generalized coordinate

Referring to the local-inertial frame of references shown in Fig. 2, we obtain the following expressions (see Upadhyaya, 1979):

Translational kinetic energy, KE_T

$$KE_T = \frac{M_f}{2}\left[\dot{v}^2 + \dot{w}^2 + \dot{u}^2 + r^2(\dot{\alpha}^2 + \dot{\beta}^2)\right] + M_f r(\dot{v}\dot{\alpha} + \dot{w}\dot{\beta}) \quad \text{[6]}$$

Rotational kinetic energy, KE_{ROT}

$$KE_{ROT} = \frac{I_f}{2}(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\psi}^2) \quad \text{[7]}$$

Potential energy due to the spring, S_p

$$S_p = \frac{S_f}{2}\left[(\alpha - v'_\ell)^2 + (\beta - w'_\ell)^2 + (\psi - \theta_\ell)^2\right] \quad \text{[8]}$$

Potential energy due to gravity, G_p

$$G_p = M_f\, g\, r\,(1 - \cos\alpha \cos\beta) \quad \text{[9]}$$

The Raleigh dissipation function P becomes

$$P = \frac{c_f}{2}\left[(\dot{\alpha} - \dot{v}'_\ell)^2 + (\dot{\beta} - \dot{w}'_\ell)^2 + (\dot{\psi} - \dot{\theta}_\ell)^2\right] \quad \text{[10]}$$

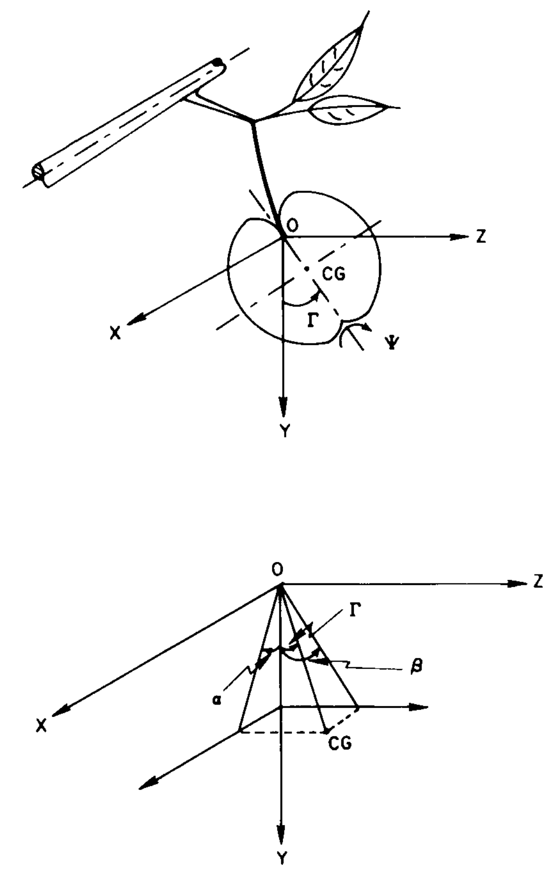

FIG. 2 Fruit-stem subsystem.

where

x,y,z = the coordinates of center of gravity of the fruit
r = distance of center of gravity of fruit from the fruit stem joint, L
v_f = translational velocity of fruit, LT^{-1}
m_f = mass of the fruit, M
I_f = moment of inertia of the fruit about its center of gravity, ML^2
g = acceleration due to gravity, LT^{-2}
c_f = torsional damping coefficient, ML^2T^{-1}
S_f = torsional spring constant, MLT^{-1}

and where dots represent differentiation with respect to time and primes represent differentiation with respect to y. The subscript ℓ stands for the displacement at the fruit end of the stem. We have Lagrangian, $\tilde{L}$

$$\tilde{L} = T - \tilde{V} = KE_T + KE_{rot} - (S_p + G_p) \quad \text{[11]}$$

T = total kinetic energy
$\tilde{V}$ = total potential energy.

Using the above relations, equations [10] and [11] in equation [5], we obtain the following six equations of motion for the fruit (Upadhyaya, 1979).

$$M_f\ddot{v} + M_f r\ddot{\alpha} = f_x \quad \text{[12a]}$$

$$M_f\ddot{u} = f_y \quad \text{[12b]}$$

$$M_f\ddot{w} + M_f r\ddot{\beta} = f_z \quad \text{[12c]}$$

$$(I_f + M_f r^2)\ddot{\alpha} + M_f r\ddot{v} + c_f(\dot{\alpha} - \dot{v}'_\ell) + S_f(\alpha - v'_\ell) + M_f g r\alpha = 0 \quad \text{[12d]}$$

$$I_f\ddot{\psi} + c_f(\dot{\psi} - \dot{\theta}) + S_f(\psi - \theta) = 0 \quad \text{[12e]}$$

$$(I_f + M_f r^2)\ddot{\beta} + M_f r\ddot{w} + c_f(\dot{\beta} - \dot{w}'_Q) + S_f(\beta - w'_Q) + M_f g r\beta = 0 \quad \text{[12f]}$$

The first three equations describe the translational motion of the fruit and the next three describe the rotational motion of the fruit.

Since the fruit is coupled elastically by torsional springs, the equations of motion of the stem must be modified to account for this elastic coupling.

The three modified beam equations (equations [1], [3] and [4]) of the motion of the stem are:

$$\rho A\ddot{v} + c_t\dot{v} + (EI\,v'')'' = c_f(\dot{\alpha} - \dot{v}')\delta'(y-L_i) + S_f(\alpha - v')\delta'(y - L_i) \quad \text{[13a]}$$

$$\rho A\ddot{w} + c_t\dot{w} + (EIw'')'' = c_f(\dot{\beta} - \dot{w}')\delta(y - L_i) + S_f(\beta - w')\delta'(y-L_i) \quad \text{[13b]}$$

$$\rho J\ddot{\theta} + c_r\dot{\theta} - (GJ\theta')' = c_f(\dot{\psi} - \dot{\theta})\delta(y - L_i) + S_f(\psi - \theta)\delta(y - L_i) \quad \text{[13c]}$$

Solution Technique, Galerkin's Criteria

The finite element method offers a convenient way of solving the above differential equations. The Galerkin method is particularly useful in this case since the forcing function is singular in its space dependence. In order to account for the deflection due to shear, the complementary energy approach is used to develop the stiffness matrix for the transverse vibrations along the x and z axes. When applying Galerkin's criterion one assumes a solution of the form:

$$\tilde{\phi} = [N(y)]\left\{\Phi(t)\right\}^e \quad \text{[14]}$$

where,

$\tilde{\phi}$ = approximate deflection along x, y or z direction about y axis

[N] = space dependent shape function, a row vector

$\{\Phi\}^e$ = time dependent displacement vector, a column vector.

Furthermore, if the differential equations are rewritten as $\hat{L}(\phi) = 0$, then in general for the approximate solution $\tilde{\phi}$, $\hat{L}(\tilde{\phi}) \neq 0$. Therefore, the residual, $\hat{e}$ is,

$$\hat{e} = \hat{L}(\tilde{\phi}) - \hat{L}(\phi) = \hat{L}(\tilde{\phi}) \quad \text{[15]}$$

where $\hat{L}$ is the differential operator and ϕ is the exact solution.

The Galerkin method minimizes this residual in the following sense

$$\int_0^{L_i} N^T\hat{L}(\tilde{\phi})dy = 0 \quad \text{[16]}$$

over each subdomain.

In order to apply the criteria we need to select approximate shape functions, N(y).

(i) for vibration along the x axis (transverse), v,

$$v = [N_1\ N_4\ N_7\ N_{10}]\begin{Bmatrix}\Phi_1\\ \Phi_4\\ \Phi_7\\ \Phi_{10}\end{Bmatrix}^e \quad \text{[17]}$$

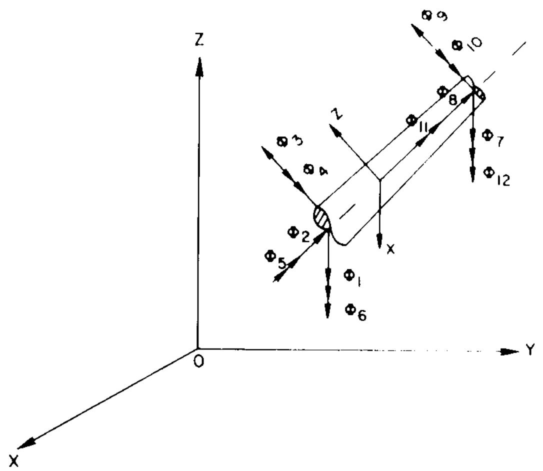

FIG. 3 Nodal degrees of freedom.

Φ_1, Φ_7 = displacements due to transverse vibration, v

Φ_2, Φ_8 = displacements due to axial vibration, u

Φ_3, Φ_9 = displacements due to transverse vibration, w

Φ_4, Φ_{10} = slopes due to transverse vibration, v

Φ_5, Φ_{11} = rotations due to torsional vibration, θ

Φ_6, Φ_{12} = slopes due to transverse vibration, w

(ii) for axial vibration, u

$$\tilde{u} = [N_2\ N_8]\begin{Bmatrix}\Phi_2\\ \Phi_8\end{Bmatrix}^e \quad \text{[18]}$$

(iii) for vibration along the z axis (transverse), w

$$\tilde{w} = [N_3\ N_6\ N_9\ N_{12}]\begin{Bmatrix}\Phi_3\\ \Phi_6\\ \Phi_9\\ \Phi_{12}\end{Bmatrix}^e \quad \text{[19]}$$

(iv) for rotational vibration, θ

$$\tilde{\theta} = [N_5\ N_{11}]\begin{Bmatrix}\Phi_5\\ \Phi_{11}\end{Bmatrix}^e \quad \text{[20]}$$

The nodal variables, ϕ's, are shown in Fig. 3. The superscript e denotes the local coordinate system associated with the element.

Since the transverse vibration equations have a fourth order derivative in space, a cubic order interpolation polynomial is selected so that the second derivative in space is continuous (Gallagher, 1975). This choice gives

$$N_1 = N_3 = 1 - 3y^2/L^2 + 2y^3/L^3$$

$$N_4 = N_6 = y - 2y^2/L + y^3/L^2$$

$$N_7 = N_9 = 3y^2/L^2 - 2y^3/L^3$$

$$N_{10} = N_{12} = -y^2/L + y^3/L^2 \quad \text{[21]}$$

For the axial and rotational vibration a linear interpolation polynomial is assumed since the respective differential equations have second derivatives in space. This gives:

$$N_2 = N_5 = 1 - y/L$$

$$N_8 = N_{11} = y/L \quad \text{[22]}$$

Having selected the appropriate shape functions we can now apply Galerkin's criterion to the differential equations to each limb and stem element.

Implementation of Galerkin's Criteria

If we assume that the element tapers linearly from the near end to the far end then the area A of the cross-section is given by:

$$A = \frac{\pi D^2}{4} = \frac{\pi}{4}(D_i + \gamma y)^2 \quad \text{[23]}$$

where,

D_i = diameter of the i^{th} node, L

y = distance from near end, L

γ = taper of the element

the taper γ is given by

$$\gamma = (D_{i+1} - D_i)/L_i \quad \text{[24]}$$

where,

D_{i+1} = diameter of the $(i+1)^{th}$ node, L

L_i = length of the i^{th} element, L

the moment of inertia I_i is given by:

$$I_i = \frac{\pi}{64}(D_i^4 + 4D_i^3\gamma y + 6D_i^2\gamma^2 y^2 + 4D_i\gamma^3 y^3 + \gamma^4 y^4) \quad \text{[25]}$$

and the polar moment of inertia J_i is given by:

$$J_i = 2I_i \quad \text{[26]}$$

The evaluation procedure is illustrated here by considering the axial vibration:

Rewriting the differential equation [2] as

$$\hat{L}(u) = \rho A u_{tt} + c_a u_t - (EAu')' - f_y[\delta(y) - \delta(y - L_i)] - F_y\delta(y-a) = 0 \quad \text{[2a]}$$

From equation [18]

$$\tilde{u} = [N_2 \; N_8]\begin{Bmatrix}\Phi_2\\ \Phi_8\end{Bmatrix}^e$$

or

$$\tilde{u} = [N]\left\{\Phi\right\}^e$$

Therefore,

$$\dot{\tilde{u}} = [N]\left\{\dot{\Phi}\right\}^e \quad \text{[18a]}$$

$$\ddot{\tilde{u}} = [N]\left\{\ddot{\Phi}\right\}^e \quad \text{[18b]}$$

$$\tilde{u}' = [N]'\left\{\Phi\right\}^e \quad \text{[18c]}$$

Applying Galerkin's criteria, we obtain

$$\int_0^{L_i} [N]^T \Big\{\rho A[N]\left\{\ddot{\Phi}\right\}^e + c_a[N]\left\{\dot{\Phi}\right\}^e - (EA[N]'\left\{\Phi\right\}^e)' - f_y[\delta(y) - \delta(y - L_i)] - F_y\,\delta(y-a)\Big\}\,dy = 0 \quad \text{[27]}$$

The first term gives the element mass matrix, the second term yields the element damping matrix, the third term gives the element stiffness matrix, the fourth term produces element nodal forces and the last term gives the external force vector for the axial vibration (Upadhyaya, 1979).

Rotational Vibration

The differential equation describing this motion is similar to axial vibration and the evaluation of element matrices proceeds similarly.

Transverse Vibrations

Computation of element matrices was much more involved and, therefore, was directly done on the computer. The element stiffness matrix was evaluated using the complementary energy approach to account for shear deflection and is outlined in Appendix A.

The terms corresponding to the external or internal moments which contain the space derivative of the Dirac delta function are evaluated as the forcing functions for the slope equations (Upadhyaya, 1979).

The application of Galerkin's criterion for each element along with the complementary energy approach leads to the element mass, damping and stiffness matrices. It evaluates the element nodal force vector as well as the element external force vector. The following matrix equation of vibration results for a vibrating limb under the action of an external force.

$$[M]^{(e)}\left\{\ddot{\Phi}\right\}^{(e)} + [C]^{(e)}\left\{\dot{\Phi}\right\}^{(e)} + [K]^{(e)}\left\{\Phi\right\}^{(e)} = \left\{f\right\}^{(e)} + \left\{F\right\}^{(e)} \quad \text{[28]}$$

where,

$[M]^{(e)}$ = element mass matrix

$[C]^{(e)}$ = element damping matrix

$[K]^{(e)}$ = element stiffness matrix

$\{f\}^{(e)}$ = element internal (nodal) force vector

$\{F\}^{(e)}$ = element external force vector.

Assemblage of Element Vibration Equations, Global Coordinates

Because each element is arbitrarily situated in space, the displacement vectors should be expressed in terms of some fixed reference coordinate system before assembling the element equations. The fixed coordinate system selected in this analysis consisted of a right handed system with positive Y axis coinciding with the axis of the clamp along a horizontal direction as shown in Fig. 4. The Z axis is directed such that the YZ plane is vertical.

The Eulerian transformations were used to obtain the equations of motion in global coordinates from those in the local coordinate system. Initially the element was assumed to be directed along the global Y axis. The element is rotated through an angle ϕ_e about the Y axis to orient it along the y′ axis of the x′y′z′ coordinate system. If we rotate the element about the x′ axis through an angle θ_e, the element will now orient itself along its natural configuration in space (Fig. 4). This transformation can be accomplished by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [\tilde{T}]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad \text{[29]}$$

where

$$[\tilde{T}] = \begin{bmatrix} \cos\phi_e & \sin\phi_e & 0 \\ -\cos\theta_e \sin\phi_e & \cos\theta_e \cos\phi_e & \sin\theta_e \\ \sin\theta_e \sin\phi_e & -\sin\theta_e \cos\phi_e & \cos\theta_e \end{bmatrix}$$

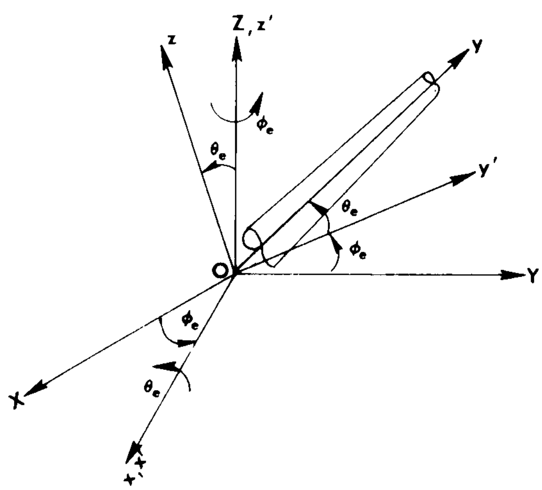

FIG. 4 Global transformation.

The element displacement vector $[\Phi]^{[e]}$ can be transformed into a global displacement vector $[\Phi]^{[g]}$ by a transformation matrix [T] such that

$$\left\{\Phi\right\}^{(e)} = [T]\left\{\Phi\right\}^{(g)} \quad [30]$$

Here [T] is a (6×6) transformation matrix.

Using the relation (equation [30]) we obtain from equation [28],

$$[M]^{(e)}[T]\left\{\ddot{\Phi}\right\}^{(g)} + [C]^{(e)}[T]\left\{\dot{\Phi}\right\}^{(g)} + [K]^{(e)}[T]\left\{\Phi\right\}^{(g)} = \left\{f\right\}^{(e)} + \left\{F\right\}^{(e)} \quad [31]$$

Since $[T]^T[T] = [I]$, we pre-multiply by $[T]^T$ to ensure symmetry and positive definiteness and obtain,

$$[M]^{(g)}\left\{\ddot{\Phi}\right\}^{(g)} + [C]^{(g)}\left\{\dot{\Phi}\right\}^{(g)} + [K]^{(g)}\left\{\Phi\right\}^{(g)} = \left\{f\right\}^{(g)} + \left\{F\right\}^{(g)} \quad [32]$$

where,

$$[M]^{(g)} = [T]^T[M]^{(e)}[T]$$

$$[C]^{(g)} = [T]^T[C]^{(e)}[T]$$

$$[K]^{(g)} = [T]^T[K]^{(e)}[T]$$

$$\left\{f\right\}^{(g)} = [T]^T\left\{f\right\}^{(e)}$$

$$\left\{F\right\}^{(g)} = [T]^T\left\{F\right\}^{(e)}$$

Forcing Function

Fig. 5 shows an impacting force arbitrarily oriented in space. If θ_f is the inclination of the force with the z' axis and ϕ_f is the inclination with respect to the y' axis in the horizontal plane, we get,

$$F_x = \overline{F}\sin\theta_f\sin\phi_f \quad [33a]$$

$$F_y = \overline{F}\sin\theta_f\cos\phi_f \quad [33b]$$

$$F_z = \overline{F}\cos\theta_f \quad [33c]$$

$$M_x = 0 \quad [33d]$$

$$M_y = -\overline{F}\hat{E}\cos\theta_f \quad [33e]$$

$$M_z = \overline{F}\hat{E}\sin\theta_f\cos\phi_f \quad [33f]$$

where,

$\hat{E}$ = eccentricity of impact, L

$\overline{F}$ = average impact force vector, MLT^{-2}

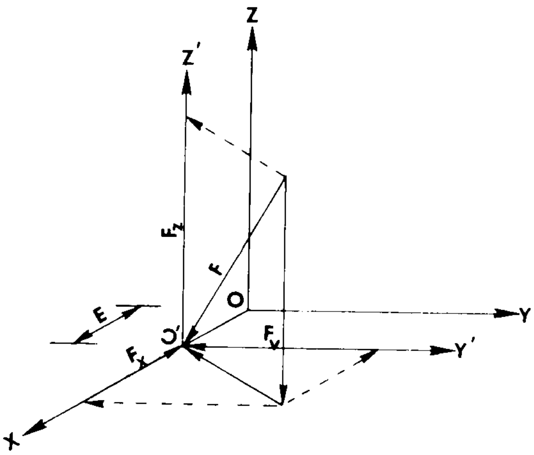

FIG. 5 Impact force.

Attachment of the Fruit

Ordinary differential equations for the motion of the fruit are placed directly in the global matrix equations for limb-stem vibration after performing the appropriate Eulerian transformation.

Transient Solution and the Newmark Method

When the element matrix equations are assembled, nodal force vectors (internal) appear in equal and opposite pairs, so they vanish. Therefore, the assembled global matrix equation is:

$$[M]^{(g)}\left\{\ddot{\Phi}\right\}^{(g)} + [C]^{(g)}\left\{\dot{\Phi}\right\}^{(g)} + [K]^{(g)}\left\{\Phi\right\}^{(g)} = \left\{F(t)\right\}^{(g)} \quad [34]$$

The transient solution is obtained using Newmark's direct integration scheme (Bathe and Wilson, 1976, p. 322-324; Upadhyaya, 1979).

In this scheme two integration parameters, $\hat{\alpha}$ and $\hat{\delta}$, are used. For $\hat{\delta} = 1/2$ and $\hat{\alpha} = 0.25(0.5 + \hat{\delta})^2$ the scheme is both accurate (e.g., does not introduce artificial damping on higher modes) and unconditionally stable (e.g., the higher frequency modes do not grow without bounds). Artificial damping can be achieved on higher modes to simulate the natural damping of higher modes using $\hat{\delta} > 0.5$.

Element Stress Resultants and Fruit Removal

It was assumed that the stem would fail in shear, as does a ductile material. The maximum shear force was calculated as follows:

If we consider the element matrix equation of vibration except for the elements where the clamp is attached, the external force vector $\{F\} = \{0\}$. Therefore,

$$[M]^{(g)}\left\{\ddot{\Phi}\right\}^{(g)} + [C]^{(g)}\left\{\dot{\Phi}\right\}^{(g)} + [K]^{(g)}\left\{\Phi\right\}^{(g)} = \left\{f\right\}^{(g)} \quad [35]$$

Once we obtain $\{\Phi\}^{[g]}$, $\{\dot{\Phi}\}^{[g]}$ and $\{\ddot{\Phi}\}^{[g]}$ we can get $\{\hat{f}\}^{[g]}$ from equation [35]. But

$$\left\{\hat{f}\right\}^{(g)} = [T]^T\left\{\hat{f}\right\}^{(e)}$$

Premultiplying by [T] and remembering $[T][T]^T = [I]$, we get

$$[T]\left\{\hat{f}\right\}^{(g)} = \left\{\hat{f}\right\}^{(e)} \quad [36]$$

where,

$$\left\{\hat{f}\right\}^{(e)} = [s_{x_i}\ f_{y_i}\ s_{z_i}\ m_{x_i}\ t_{y_i}\ m_{z_i}\ s_{x_{i+1}}\ f_{y_{i+1}}\ s_{z_{i+1}}\ m_{x_{i+1}}$$
$$t_{y_{i+1}}\ m_{x_{i+1}}]^T$$

s_x, s_z = shear force at a given node in x and z direction, MLT^{-1}
m_x, m_z = bending moment at a given node in x and z direction, ML^2T^{-2}
f_y = axial force at a given point, MLT^{-2}
t_y = torque at a given point, ML^2T^{-2}

At the i^{th} node the total bending moment is,

$$m_t = [m_{x_i}^{\ 2} + m_{z_i}^{\ 2}]^{1/2} \quad \text{[37]}$$

The bending stress from flexure theory is:

$$\sigma = \frac{m_t\, D_i}{2\, I_i}\ (\text{Since } y = \frac{D_i}{2}) \quad \text{[38]}$$

The direct tensile stress is f_{y_i}/A_i

The total tensile stress,

$$f_t = \frac{f_{y_i}}{A_i} \pm \frac{m_t\, D_i}{2\, I_i} \quad \text{[39]}$$

The total shear stress from the torsion equation is:

$$f_s = \frac{t_{y_i}\, D_i}{2J} \quad \text{[40]}$$

The maximum shear stress is

$$(f_s)_{max} = \frac{1}{2}\, [f_t^{\ 2} + 4 f_s^{\ 2}]^{1/2} \quad \text{[41]}$$

Computer Program for the Limb Impact Problem

The solution to the finite element model was obtained using a PL/I computer program on an IBM 370/168. The program required a region of 240K and 19 s of CPU time to compile. The I/0 time was 150 s. For a limb of 30 elements about 150K region of memory and 21 s of CPU time was required to execute. The I/0 time was 3 s.

MODEL VERIFICATION

Implementation of Newmark's direct integration scheme was verified by solving example 8.4 on page 324-325 of Bathe and Wilson (1976). The results from the program agreed with those given in the book. The entire program was checked for accuracy by considering the static deflections of a beam. The output of the model agreed with the theoretical results. See Part II for a discussion of the experimental verification of the model.

CONCLUSIONS

1 We have developed a finite element model to predict the dynamic behavior of a fruit bearing limb impacted at its base. This model was verified to be quite accurate when compared with simpler test cases and also with the experimental results.

2 We found during the course of this study that the use of Timoshenko beam theory with the finite element method can lead to ill-conditioned matrices if the shear energy associated with the problem is small. In such cases the complementary energy approach was found to be useful.

APPENDIX A

Element Stiffness Matrix for Transverse Vibrations

Gallagher (1975, p. 170-173) gives complementary energy, π_b, in bending as:

$$\pi_b = \frac{1}{2} \int_{V_e} \sigma [E]^{-1} \sigma dV_e - \int_{S_e} \overline{T} \cdot \overline{u}\ dS_e \quad \text{[A.1]}$$

where,
σ = stress distribution, $ML^{-1}T^{-2}$
S_e = the surface over which the displacements are prescribed, L^2
$\overline{u}$ = prescribed displacement, L
$\overline{T}$ = traction force, MLT^{-2}
[E] = elastic stiffness matrix

For the beam element we have from flexure theory:

$$\sigma = \frac{\hat{M} y}{I} \quad \text{[A.2]}$$

where

$\hat{M}$ = moment at a given section, ML^2T^{-2}
y = distance from neutral axis, L
I = area moment of inertia, L^4

Therefore for beam elements equation [A.1] becomes

$$\pi_b = \frac{1}{2} \int_{V_e} (\frac{\hat{M} y}{I}\ \frac{1}{E}\ \frac{\hat{M} y}{I} dA dy) - [f_i\ m_i\ f_{i+1}\ m_{i+1}] \left\{\Phi\right\}^e \quad \text{[A.3]}$$

Since

$\int_A y^2\, dA = I,$ we have,

$$\pi_b = \frac{1}{2} \int_o^{L_i} \frac{\hat{M}^2}{EI} dy - [f_i\ m_i\ f_{i+1}\ m_{i+1}] \left\{\Phi\right\}^e \quad \text{[A.4]}$$

From Figure A.1

$$\hat{M} = y f_i + m_i = [y\ \ 1] \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} \quad \text{[A.5]}$$

Therefore,

$$\pi_b = \frac{1}{2} \int_o^{L_i} \frac{[f_i\ m_i]}{EI} \begin{bmatrix} y^2 & y \\ y & 1 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} dy$$
$$- [f_i\ m_i\ f_{i+1}\ m_{i+1}] \left\{\phi\right\}^e \quad \text{[A.6]}$$

In order to account for shear deflection, the shear energy due to bending is introduced into the above expression. Shear energy due to bending is given by Gallagher (1975, p. 366) as:

$$\pi_s = \frac{1}{2} \int_o^{L_i} (\gamma_s)^2\, K_S\, A\, G\, dy \quad \text{[A.7]}$$

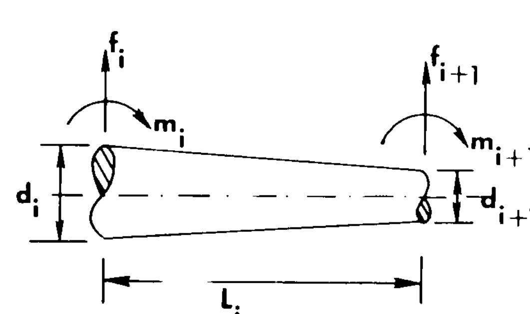

FIG. A.1 Force diagram for beam element.

where,

γ_s = shear deflection

G = shear modulus, $ML^{-1}T^{-2}$

K_s = shear coefficient

But,

$$\gamma_S = \frac{f_i}{K_S AG} = \frac{1}{K_S AG} [1\ 0] \begin{Bmatrix} f_i \\ m_i \end{Bmatrix}$$

therefore,

$$\pi_S = \int_o^{L_i} \left(\frac{1}{K_S AG}\right) [f_i\ m_i] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} dy \quad \ldots\ldots\ldots [A.8]$$

Adding this to the complementary energy expression, we get,

$$\pi_c = \frac{1}{2} \int_o^{L_i} [f_i\ m_i] \begin{bmatrix} y^2 & y \\ y & 1 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} dy + \frac{1}{2} \int_o^{L_i} \left(\frac{1}{K_S GA}\right) [f_i\ m_i] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} dy - [f_i\ m_i\ f_{i+1}\ m_{i+1}] \{\Phi\}^e \quad \ldots\ldots [A.9]$$

From equilibrium considerations,

$$f_i = -f_{i+1}$$

$$m_i + f_i L_i = -m_{i+1}$$

$$\begin{Bmatrix} f_{i+1} \\ m_{i+1} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ -L_i & -1 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = [R]_S \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} \quad \ldots\ldots\ldots [A.10]$$

where,

$$[R]_S = \begin{bmatrix} -1 & 0 \\ -L_i & -1 \end{bmatrix}$$

$$\begin{Bmatrix} f_i \\ m_i \\ f_{i+1} \\ m_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ -L_i & -1 \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = \begin{bmatrix} [I] \\ [R]_S \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix}$$

$$[f_i\ m_i\ f_{i+1}\ m_{i+1}] = [f_i\ m_i]\ [[I]\ [R]_S^T]$$

$$\pi_c = \frac{1}{2} \int_o^{L_i} [f_i\ m_i] \begin{bmatrix} \frac{y^2}{EI} + \frac{1}{K_S GA} & \frac{y}{EI} \\ \frac{y}{EI} & \frac{1}{EI} \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} dy - [f_i\ m_i]\ [[I]\ [R]_S^T]$$

$$\frac{\partial \pi_c}{\partial \{\bar{f}\}} = \int_o^{L_i} \begin{bmatrix} \frac{y^2}{EI} + \frac{1}{K_S GA} & \frac{y}{EI} \\ \frac{y}{EI} & \frac{1}{EI} \end{bmatrix} dy \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} - [[I]\ [R]_S^T] \{\Phi\}^e = 0$$

where

$$\{\bar{f}\} = \begin{Bmatrix} f_i \\ m_i \end{Bmatrix}$$

Therefore,

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = [[I]\ [R]_S^T] \{\Phi\}^e$$

$$[f] \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = [[I]\ [R]_S^T] \{\Phi\}^e \quad \ldots\ldots\ldots\ldots\ldots\ldots [A.11]$$

$$f_{11} = \int_o^{L_i} \left(\frac{y^2}{EI} + \frac{1}{K_S GA}\right) dy$$

$$f_{12} = f_{21} = \int_o^{L_i} (y/EI) dy$$

$$f_{22} = \int_o^{L_i} \frac{dy}{EI}$$

From equation [A.11]

$$\begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = [f]^{-1}\ [[I]\ [R]_S^T] \{\Phi\}^e$$

Multiplying both sides by $\begin{bmatrix} [I] \\ [R]_S \end{bmatrix}$ we get

$$\begin{bmatrix} [I] \\ [R]_S \end{bmatrix} \begin{Bmatrix} f_i \\ m_i \end{Bmatrix} = \begin{bmatrix} [I] \\ [R]_S \end{bmatrix} [f]^{-1}\ [[I]\ [R]_S^T] \{\Phi\}^e$$

$$\begin{Bmatrix} f_i \\ m_i \\ f_{i+1} \\ m_{i+1} \end{Bmatrix} = [K] \{\Phi\}^e$$

where

$$[K] = \begin{bmatrix} [I] \\ [R]_S \end{bmatrix} [f]^{-1}\ [[I]\ [R]_S^T] = \begin{bmatrix} [f]^{-1} & [f]^{-1} [R]_S^T \\ [R]_S [f]^{-1} & [R]_S [f]^{-1} [R]_S^T \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots [A.12]$$

The stiffness matrix given by equation [A.12] is used in our analysis.