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BIFURCATION OF 4:1 SUBHARMONICS IN THE NONLINEAR MATHIEU EQUATION

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Introduction

It is well known [1] that the linear Mathieu equation,

$$\ddot{u} + (\delta + \varepsilon \cos t)\, u = 0 \tag{1.1}$$

exhibits 2:1 subharmonics for δ near 1/4 when ε is small. Specifically these periodic motions with frequency 1/2 occur on the transition curves in the δ-ε plane separating regions of stability from regions of instability. Here instability refers to unbounded behavior of motions starting arbitrarily close to the zero solution $u(t) \equiv 0$. (Although eq. (1.1) exhibits other instabilities of the zero solution for δ near $n^2/4$ ($n = 2,3,4,\ldots$), none of these are subharmonics; for $n > 2$ they are superharmonics.)

Several recent works [2], [3], [4] have considered the effects of a nonlinear term on this 2:1 subharmonic by investigating the nonlinear Mathieu equation,

$$\ddot{u} + (\delta + \varepsilon \cos t)u + \varepsilon\alpha u^3 = 0\ . \tag{1.2}$$

It was shown that the nonlinearity stabilizes the unstable motion when $\delta \approx 1/4$, $\varepsilon \ll 1$, in the sense that for those values of δ,ε for which (1.1) exhibits unbounded motions, all motions of (1.2) are bounded. For such δ,ε the zero solution of (1.2) is locally unstable but globally stable; the 2:1 resonance is balanced by a shift in frequency caused by the nonlinearity, resulting in a finite amplitude 2:1 subharmonic. See Fig. 1.

The purpose of this paper is to investigate the nature of 4:1 subharmonics in eq. (1.2), and to compare their behavior with the 2:1 subharmonics studied previously.

Lie Transforms

The following exposition of Lie transforms, a perturbation method for Hamiltonian systems, is based on the work of Deprit [5] and Kamel [6].

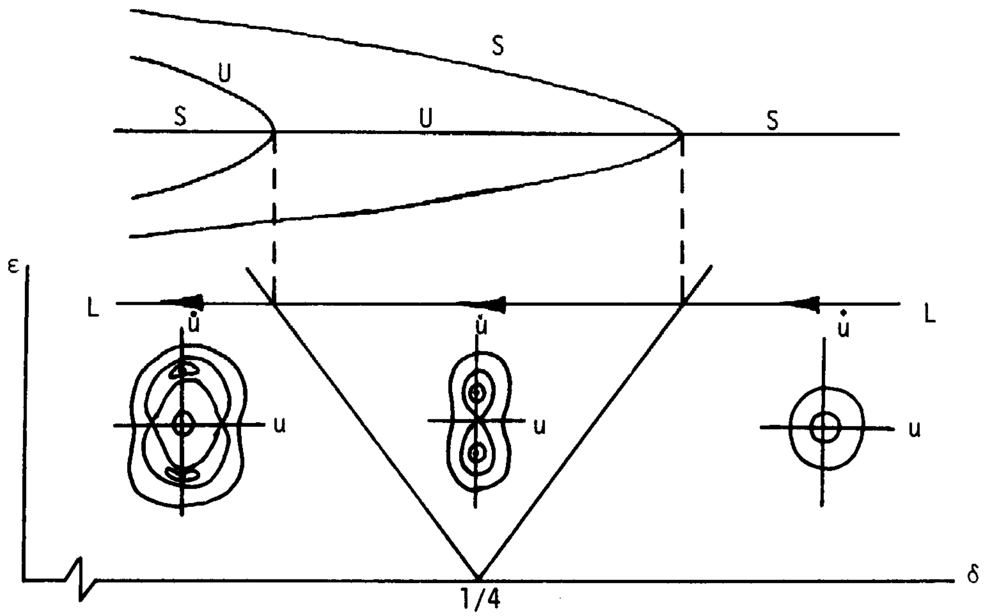

FIG. 1

Bifurcation of 2:1 subharmonics in the nonlinear Mathieu equation (1.2), after [4]. Transition curves in the δ-ε plane are $\delta = 1/4 \pm \varepsilon/2 + O(\varepsilon^2)$. In each of 3 distinct regions sketches are shown of invariant curves in the Poincaré map corresponding to surface of section $t = 0 \pmod{2\pi}$. Quasistatic motion along line LL produces the double pitchfork bifurcation diagram shown at the top of the Figure. Points on the bifurcation diagram correspond to singular points in the Poincaré map. The prongs of the pitchfork correspond to 2:1 subharmonic periodic motions of (1.2), while the handle corresponds to the zero solution. S = stable, U = unstable.

Given a Hamiltonian which is dependent upon a small parameter ε,

$$H(x,X,t,\varepsilon) \quad , \tag{2.1}$$

it is desired to obtain approximate expressions for $x(t,\varepsilon)$, $X(t,\varepsilon)$, where

$$\dot{x} = H_X \ , \ \dot{X} = -H_x \tag{2.2}$$

and where subscripts represent partial differentiation.

The method involves developing a near identity canonical transformation to new variables y,Y based on the generating equations

$$x' = W_X \ , \ X' = -W_x \tag{2.3}$$

where primes represent differentiation with respect to ε, and where

$$W(x,X,t,\varepsilon) = W_1 + \varepsilon W_2 + O(\varepsilon^2) \tag{2.4}$$

is a generating function to be chosen at our discretion. Eqs. (2.3) have the following initial conditions (at $\varepsilon = 0$):

$$x(t,0) = y(t) \ , \quad X(t,0) = Y(t) \quad . \tag{2.5}$$

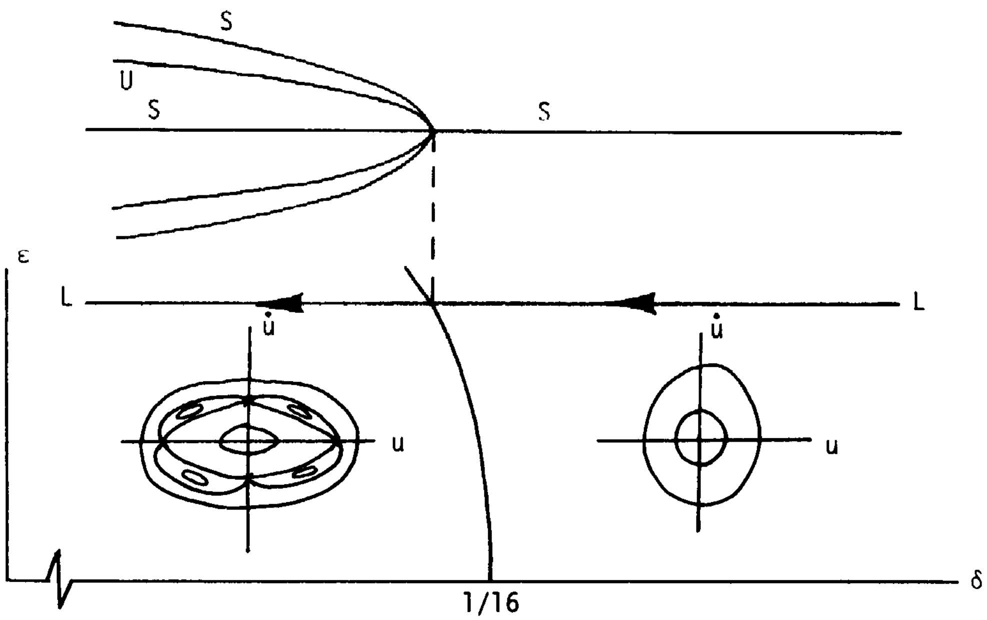

FIG. 2

Bifurcation of 4:1 subharmonics in the nonlinear Mathieu eq. (1.2) for $\alpha, \varepsilon > 0$. Transition curve is given by eq. (3.21). Quasistatic motion along line LL produces the pitchfork bifurcation diagram shown at the top of the Figure. The prongs of the pitchfork correspond to 4:1 subharmonic periodic motions of (1.2), while the handle corresponds to the zero solution. S = stable, U = unstable. Cf. Fig. 1.

The new variables y, Y are called the Lie transforms of x, X, respectively. It has been shown that this transformation is canonical [5], [6], and therefore that y, Y satisfy Hamiltonian equations:

$$\dot{y} = K_Y \ , \quad \dot{Y} = -K_y \tag{2.6}$$

where the transformed Hamiltonian K is given by the generating equation

$$H' = W_t \tag{2.7}$$

with the initial condition (at $\varepsilon = 0$)

$$H(x,X,t,0) = K(y,Y,t) \quad . \tag{2.8}$$

Note that in (2.8) K is a function of the new variables since $x = y$ and $X = Y$ at $\varepsilon = 0$. Note also that if the functions y, Y and K of eqs. (2.5), (2.8) were to depend upon an independent parameter, ε_1 say, the canonical property of the transformation would be preserved. In particular we may choose $\varepsilon_1 = \varepsilon$ and write

$$y(t,\varepsilon) \ , \quad Y(t,\varepsilon) \quad \text{and} \quad K(y,Y,t,\varepsilon) \quad . \tag{2.9}$$

The formal method is derived by constructing approximate solutions to (2.3), (2.7) via Taylor series. E.g. let us expand $x(t,\varepsilon)$ in a Taylor series about

$\varepsilon = 0$,

$$x(t,\varepsilon) = x(t,0) + \varepsilon x'(t,0) + \varepsilon^2 x''(t,0)/2 + O(\varepsilon^3) \quad . \tag{2.10}$$

From (2.3) - (2.5),

$$x'(t,0) = W_X(x,X,t,0) = W_{1Y}(y,Y,t) \quad . \tag{2.11}$$

Differentiating the first eq. of (2.3) with respect to ε,

$$x''(t,\varepsilon) = W_{xX}\, x' + W_{XX}\, X' + W_{\varepsilon X} \tag{2.12}$$

Using (2.3) - (2.5) this gives

$$x''(t,0) = W_{1yY}\, W_{1Y} - W_{1YY}\, W_{1y} + W_{2Y} \quad . \tag{2.13}$$

To condense the notation we represent the Poisson bracket (or Lie derivative) of two functions f and g by

$$(f;g) = f_y\, g_Y - f_Y\, g_y \quad . \tag{2.14}$$

Then (2.10) becomes

$$x = y + \varepsilon\, W_{1Y} + (\varepsilon^2/2)[(W_{1Y};W_1) + W_{2Y}] + O(\varepsilon^3) \tag{2.15}$$

Similarly we find

$$X = Y - \varepsilon\, W_{1y} - (\varepsilon^2/2)[(W_{1y};W_1) + W_{2y}] + O(\varepsilon^3) \tag{2.16}$$

$$H = K + \varepsilon\, W_{1t} + (\varepsilon^2/2)[(W_{1t};W_1) + W_{2t}] + O(\varepsilon^3) \tag{2.17}$$

On the other hand H may also be expanded in a power series in ε by writing

$$H(x,X,t,\varepsilon) = H_0(x,X,t) + \varepsilon\, H_1(x,X,t) + \varepsilon^2\, H_2(x,X,t)/2 + O(\varepsilon^3) \tag{2.18}$$

Here the given functions H_0, H_1, H_2 may themselves be expanded in Taylor series for small ε (since x,X depend on ε), yielding

$$H_0(x,X,t) = H_0(y,Y,t) + \varepsilon\,(H_0;W_1) + (\varepsilon^2/2)[((H_0;W_1);W_1) + (H_0;W_2)] + O(\varepsilon^3) \tag{2.19}$$

$$H_1(x,X,t) = H_1(y,Y,t) + \varepsilon(H_1;W_1) + O(\varepsilon^2) \tag{2.20}$$

$$H_2(x,X,t) = H_2(y,Y,t) + O(\varepsilon) \tag{2.21}$$

Finally we can combine (2.17) - (2.21) to obtain the following expression for $K(y,Y,t,\varepsilon)$:

$$K = K_0 + \varepsilon\, K_1 + \varepsilon^2\, K_2/2 + O(\varepsilon^3) \tag{2.22}$$

where

$$K_0 = H_0 \tag{2.23}$$

$$K_1 = H_1 + (H_0;W_1) - W_{1t} \tag{2.24}$$

$$K_2 = H_2 + (H_0;W_2) + 2(H_1;W_1) + ((H_0;W_1);W_1) - (W_{1t};W_1) - W_{2t} \tag{2.25}$$

Eq. (2.25) can be simplified using (2.24) to give

$$K_2 = H_2 + (H_1 + K_1;W_1) + (H_0;W_2) - W_{2t} \tag{2.26}$$

Note that in (2.23) - (2.26) H_0, H_1, H_2, W_1, W_2 are functions of y,Y,t.

<u>Application To Nonlinear Mathieu Equation</u>

The Hamiltonian for eq. (1.2) is

$$H = \dot{u}^2/2 + (\delta + \varepsilon \cos t)\, u^2/2 + \varepsilon\, \alpha\, u^4/4 \quad . \tag{3.1}$$

In order to investigate 4:1 subharmonics we set

$$\delta = 1/16 + \varepsilon\,\delta_1 + \varepsilon^2\,\delta_2 + O(\varepsilon^3) \quad . \tag{3.2}$$

We begin by transforming to action-angle variables x,X for the $\varepsilon = 0$ problem,

$$\dot{u} = \sqrt{X/2}\,\cos x\ , \quad u = \sqrt{8X}\,\sin x \quad . \tag{3.3}$$

Substitution of (3.3) into (3.1) puts H in the form of (2.18) with

$$H_0 = X/4 \tag{3.4}$$

$$H_1 = 4X(\delta_1 + \cos t)\sin^2 x + 16\alpha\, X^2 \sin^4 x \tag{3.5}$$

$$H_2 = 8X\,\delta_2 \sin^2 x \quad . \tag{3.6}$$

Next we perform the Lie transform described in section 2. We replace x,X by y,Y as in (2.15),(2.16). The new Hamiltonian $K(y,Y,t,\varepsilon)$ is given by (2.22) with

$$K_0 = H_0 = Y/4 \quad . \tag{3.7}$$

K_1 is obtained from eq. (2.24), which becomes

$$K_1 = H_1 - L\,W_1 \tag{3.8}$$

where $L = \frac{1}{4}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$

and where H_1 is given by (3.5) with x,X replaced by y,Y. We choose W_1 so as to eliminate all angle terms y from K_1. This requires that

$$W_1 = 2Y[\sin t - 2(\delta_1 + 4\alpha Y)\sin 2y + \alpha Y \sin 4y + \sin(2y - t) - (1/3)\sin(2y + t)] \tag{3.9}$$

whereupon $K_1 = 2\delta_1 Y + 6\alpha Y^2 \quad .$ (3.10)

Similarly, K_2 is found from (2.26),

$$K_2 = H_2 + (H_1 + K_1 ; W_1) - L\,W_2 \tag{3.11}$$

Again we try to choose W_2 so as to eliminate all the angle terms y from K_2. This time, however, it turns out that the right hand side of (3.11) contains a term of the form cos (4y - t) which lies in the null space of the operator L and hence cannot be removed from K_2. We find that

$$K_2 = 4Y[\delta_2 + \tfrac{2}{3} - 4\delta_1{}^2 - 48\delta_1\alpha Y - 136\alpha^2 Y^2 + 4\alpha Y \cos(4y - t)] \tag{3.12}$$

In order to obtain an approximate first integral we make a final canonical transformation to ψ,J defined by

$$\psi = y - t/4\ , \quad J = Y \quad . \tag{3.13}$$

This transformation is based on the generating function

$$F(y,J,t) = J(y - t/4) \tag{3.14}$$

and hence the new Hamiltonian $\tilde{K}$ is given by

$$\tilde{K} = K + F_t = K - J/4 \tag{3.15}$$

$$\tilde{K} = \varepsilon[2J(\delta_1 + 3\alpha J)] + \varepsilon^2[2J(\delta_2 + \tfrac{2}{3} - 4\delta_1{}^2 - 48\delta_1\alpha J - 136\alpha^2 J^2 + 4\alpha J \cos 4\psi)] + O(\varepsilon^3) \tag{3.16}$$

Note that $\tilde{K}$ does not depend explicitly on t and hence is a first integral of the system

$$\dot{\psi} = \tilde{K}_J \ , \quad \dot{J} = -\tilde{K}_\psi \ . \tag{3.17}$$

The singular points of (3.17) correspond via (3.13), (2.15), (2.16), (3.3) to 4:1 subharmonic periodic motions of (1.2). To find them we require $\dot{J}$ and $\dot{\psi}$ to vanish.

$$\dot{J} = 0 \Rightarrow \text{Either } J = 0 \text{ or } \sin 4\psi = 0 \ , \ \psi = n\pi/4 \ . \tag{3.18}$$

We ignore $J=0$ which corresponds to the zero solution of (1.2) and substitute $\psi = n\pi/4$ into

$$\begin{aligned}\dot{\psi} = 0 \Rightarrow \delta_1 + 6\alpha J + \varepsilon[\delta_2 + \tfrac{2}{3} - 4\delta_1{}^2 - 96\delta_1\alpha J - 408\alpha^2 J^2 \\ + 8\alpha(-1)^n J] + O(\varepsilon^3) = 0\end{aligned} \tag{3.19}$$

Eq. (3.19) is a quadratic on J. For given $\psi = n\pi/4$, one of the roots of (3.19) turns out to be of $O(1/\varepsilon)$. We omit consideration of such singular points coming from infinity since our perturbation method requires J to be O(1). The other root of (3.19) may be written in the form

$$J = \frac{-1}{6\alpha}\{\delta_1 + \varepsilon[\delta_2 + \tfrac{2}{3} + \tfrac{2}{3}\delta_1{}^2 - \tfrac{4}{3}(-1)^n\delta_1]\} + O(\varepsilon^2) \tag{3.20}$$

Now for real solutions u(t), J must be non-negative (cf. (3.3)). For $\alpha,\varepsilon > 0$ this requires to O(1) that $\delta_1 \leq 0$. In the transition case $\delta_1 = 0$, it is required to $O(\varepsilon)$ that $\delta_2 + 2/3 \leq 0$. This shows that there is a transition curve in the δ-ε plane given by

$$\delta = 1/16 - (2/3)\varepsilon^2 + O(\varepsilon^3) \ . \tag{3.21}$$

For $\alpha,\varepsilon > 0$ points lying to the left of this curve exhibit two 4:1 subharmonics, corresponding respectively to n odd and even. Linearization of (3.17) about the singular points (3.18),(3.20) determines their stability: we find instability for n even ($\psi = 0$, $\pi/2$, π , $3\pi/2$) and stability for n odd ($\psi = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$).

These results may be pictured by transforming back to $u,\dot{u},t$. We omit the algebra and sketch the results of setting t = 0 in the first integral corresponding to (3.16). This gives the invariant curves in an approximation to the Poincaré map corresponding to the surface of section t = 0 (mod 2π). See Fig. 2.

Discussion

In this section we compare the behavior of the 4:1 subharmonics derived in section 3 with the 2:1 subharmonics studied previously.

Let us compare Figs. 1 and 2. While the 2:1 subharmonic in Fig. 1 has a region of local instability of the zero solution associated with it, no such region occurs in Fig. 2. The origin remains locally stable as one crosses the transition curve in Fig. 2. Moreover the bifurcations of each of the two 2:1 subharmonics present in the left-most region of Fig. 1 occur at distinct points (as one crosses each of the two transition curves). In Fig. 2, however, both 4:1 subharmonics bifurcate simultaneously.

The relationship between these two bifurcations can be better understood by considering the linear Mathieu eq. (1.1). As noted in the Introduction, (1.1) has 2:1 subharmonics occurring on each of the two transition curves in Fig. 1. We now show, however, that (1.1) has two linearly independent 4:1 subharmonics occurring on the single transition curve of Fig. 2. This follows from (3.16), (3.17) with $\alpha = 0$, which give

$$J = J_0 \ , \ \psi = \omega t + \psi_0 \tag{4.1}$$

where $$\omega = \tilde{K}_J = 2\varepsilon[\delta_1 + \varepsilon(\delta_2 + \frac{2}{3} - 4\delta_1^{\,2})] \ , \tag{4.2}$$

whereupon, neglecting terms of $O(\varepsilon)$,

$$u(t) = \sqrt{8J_0} \ \sin \ [(\omega + \tfrac{1}{4})t + \psi_0] \tag{4.3}$$

In order that (4.3) represent a 4:1 subharmonic we require that

$$\omega = 0 \Rightarrow \delta_1 = 0 \ , \ \delta_2 = -2/3 \ , \tag{4.4}$$

i.e. we are on the transition curve (3.21).

The coexistence of two linearly independent 4:1 subharmonics on the transition curve of Fig. 2 suggests that we consider the bifurcation of Fig. 2 to be a limiting case of the bifurcation of Fig. 1 in which the region of instability of Fig. 1 becomes smaller and finally vanishes as the two transition curves are brought closer together and finally coalesce.

We also note that the analysis leading to Fig. 1 neglected terms of $O(\varepsilon^2)$ while the analysis in this paper leading to Fig. 2 necessarily had to keep $O(\varepsilon^2)$ terms. This suggests that the 2:1 subharmonics are more prominent for small ε than the 4:1 subharmonics. To test this we numerically integrated eq. (1.2) and generated Poincaré maps as in Figs. 1,2. We found that when $\delta \approx 1/16$ (to the left of the transition curve (3.21)) both the 4:1 subharmonics of Fig. 2 and the 2:1 subharmonics of Fig. 1 were present. However the 2:1 occurred for larger amplitudes than the 4:1, and the region enclosed by the separatrices was much larger for the 2:1 than for the 4:1. We conjecture that the additional singular point of eq. (3.19) which was of $O(1/\varepsilon)$ represents the 2:1 subharmonic. Since the latter occurs at a finite amplitude for

$\delta \approx 1/16$ we are not surprised that the perturbation method sees it as coming from infinity.

Finally we note that all the results of this paper are only valid for small ε, not only because the perturbation method requires it, but also because KAM theory tells us that chaos will replace the invariant tori of Figs. 1,2 as ε is increased [7].

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