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In Memoriam P. D. Panagiotopoulos (1950 - 1998)

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Relaxing Nonholonomic Constraints

Richard Rand and Deepak Ramani [1]

Introduction

A constraint is a geometric restriction on the motion of a dynamical system. It is enforced by the application of one or more constraint forces. Instead of being prescribed, the constraint forces take on whatever values are necessary to produce the constrained motion.

It is often desirable to be able to relax a given constraint without eliminating it altogether. For example, the rolling-without-slipping of a rigid body on a plane is well known to be a nonholonomic constraint. By relaxing such a constraint, one could study the effect of rolling with limited slipping.

The idea of this work is to provide a one-parameter family of systems which serve to relax a given constraint. This is accomplished by replacing the constraint force by a force directed in the same direction, but with a value which is a fractional multiple $0 < \lambda < 1$ of the value necessary to enforce the given constraint.

We illustrate the idea by applying it to an example based on the motion of a supercavitating underwater projectile [3]. The presence of tail fins on such a body suggests a nonholonomic constraint which requires the velocity of the tail to be directed along the axis of the body, similar to an arrow in flight. This idealized constraint is unattainable, however, and a more realistic situation may be achieved by relaxing this nonholonomic constraint as described above.

Kiceniuk Forces

When a body moves through water at sufficient speed, the fluid pressure may drop locally below a level which sustains the liquid phase, and a low-density gaseous cavity can form. Flows exhibiting cavities enveloping a moving body entirely are called *supercavitating,* and, since the liquid phase does not contact the moving body through most of its length, skin drag is almost negligible. In [1], Kiceniuk experimentally measured lift and drag forces on a supercavitating body. As in [3], we idealize his data by assuming that (i) the net force F acts along the axis of the body, (ii) the net moment

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by the fluid on the body is zero, and (iii) the magnitude of F is given by the expression:

$$F = \frac{1}{2}\,\rho\, A\, k\, s^2\, \cos\alpha \tag{1}$$

where $s =$ speed,
α is the angle between the body's velocity vector and its symmetry axis,
$\rho =$ density of water,
$A =$ cross-sectional area at the body's tip,
$k =$ a nondimensional constant.

Equations of Motion

A supercavitating projectile with fins may be modeled by assuming that the fins do not permit motion perpendicular to the projectile's axis at the point where the fins are attached. This nonholonomic constraint has been called *Caplygin's sleigh* [2]. The equations of plane motion are (see Fig.1):

$$F_c = m\,(\dot{v} + \omega\, u + a\,\dot{\omega}) \tag{2}$$

$$-F = m\,(\dot{u} - \omega\, v - a\,\omega^2) \tag{3}$$

$$I\,\dot{\omega} = -F_c\, a \tag{4}$$

where u=projectile velocity in axial direction,
v=projectile velocity at fin attachment point in direction normal to axis,
ω=angular velocity of projectile,
F_c=constraint force,
F=Kiceniuk force of eq.(1),
a=distance from center of mass to fin attachment point,
m=mass of projectile,
I=moment of inertia about center of mass.

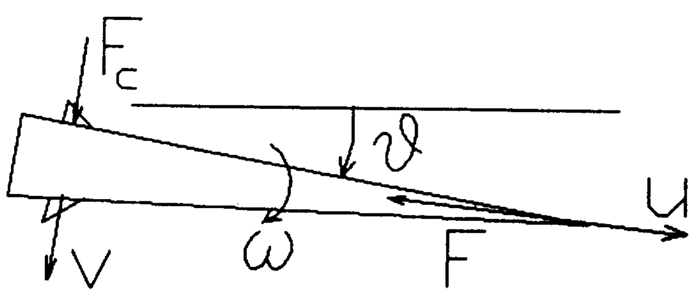

Fig.1. A supercavitating projectile with fins.

If the fins were perfectly effective in administering the constraint $v = 0$, the associated constraint force F_c could be obtained by setting $v = 0$ in eq.(2) and substituting $\dot{\omega}$ from eq.(4):

$$F_c = \frac{mI}{I + ma^2}\, u\, \omega \tag{5}$$

However, the fins are expected to be rather ineffective since in the supercavitated state they interact only with the water vapor inside the cavity rather than with the water itself. As explained in the introduction, we will model the relaxed constraint by replacing F_c by λ times the value given by eq.(5):

$$F_c = \lambda \, \frac{mI}{I + ma^2} \, u \, \omega, \qquad 0 < \lambda < 1 \tag{6}$$

The expression (1) for the Kiceniuk force F can be rewritten in terms of the state variables u, v, ω:

$$F = \beta \, m \, u \, \sqrt{u^2 + (v + a\omega)^2} \tag{7}$$

where $\beta = \frac{\rho A k}{2m}$. When expressions (6) and (7) are substituted into eqs.(2)-(4), we obtain:

$$\dot{v} = (\lambda - 1) \, u \, \omega \tag{8}$$

$$\dot{u} = a \, \omega^2 + \omega \, v - \beta \, u \, \sqrt{u^2 + (v + a\omega)^2} \tag{9}$$

$$\dot{\omega} = -\frac{\lambda J}{a} \, u \, \omega \tag{10}$$

where $J = \frac{ma^2}{I + ma^2}$. In addition to modeling the relaxed constraint force by eq.(6), our approach is to also require that the initial conditions be consistent with the constraint. In this case we require that the projectile have no sideways velocity v at the fin attachment point:

$$v(0) = 0 \tag{11}$$

Eqs.(8)-(11) are the governing equations for the system with relaxed constraint. They include the two special cases $\lambda = 1$ (the original unrelaxed constraint) and $\lambda = 0$ (the unconstrained problem).

Stability of Motion

Eqs.(8)-(11) admit the exact solution:

$$u(t) = \frac{U_0}{1 + \beta U_0 t}, \qquad v(t) = 0, \qquad \omega(t) = 0. \tag{12}$$

We investigate the stability of this motion due to a small initial angular velocity ω_0:

$$u(0) = U_0, \qquad v(0) = 0, \qquad \omega(0) = \omega_0 \tag{13}$$

Linearizing eqs.(8)-(10) about the solution (12) for the initial conditions (13) leaves $u(t)$ unchanged and yields the following expressions for the disturbed motions $v(t)$ and $\omega(t)$:

$$v(t) = \frac{a(1 - \lambda)}{J\lambda} [(1 + \beta U_0 t)^{-p} - 1] \, \omega_0 \tag{14}$$

$$\omega(t) = \omega_0 \ (1 + \beta U_0 t)^{-p} \tag{15}$$

where $p = \frac{J\lambda}{a\beta}$. In the special case $\lambda = 0$, these become:

$$v(t) = -\frac{1}{\beta} \ln(1 + \beta U_0 t)\, \omega_0, \qquad \omega(t) = \omega_0 \tag{16}$$

The displacements x, y of the attachment point on the fin relative to a nonrotating frame, and the rotation angle θ may be obtained from:

$$\dot{x} = u\cos\theta - v\sin\theta, \qquad \dot{y} = -u\sin\theta - v\cos\theta, \qquad \dot{\theta} = \omega. \tag{17}$$

We numerically integrated eqs.(17) for $\omega_0 = 0.5\ rad/sec$, using the parameters given in [3]:

$$U_0 = 1500\ m/sec, \quad a = 0.06\ m, \quad \beta = 0.00942\ m^{-1}, \quad J = 0.8823. \tag{18}$$

The resulting trajectories for one second of flight are shown in Fig.2 for $\lambda = 0,\ 0.001,\ 0.01$ and 1. Note that although the perfectly constrained motion ($\lambda = 1$) completely eliminates the instability, even a relatively small value of λ has a large stabilizing effect.

Fig.2. Trajectories after 1 second of flight for (top to bottom) $\lambda = 0,\ 0.001,\ 0.01,\ 1$. For $\lambda = 1$, the horizontal distance traveled is 288 m.

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