On the Stability of a Differential Equation With Application to Parametrically Excited Systems

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The stability of the equation $\ddot{z} + (\Delta + \epsilon \cos t)^{-m}z = 0$, where m is a positive integer, is studied by using Floquet theory and perturbations. The results are confirmed by a digital computer program based on Floquet theory. Physical examples involving parametric excitation for $m = 1, 3$ are cited from the literature.

Introduction

THE differential equation

$$\ddot{z} + (\Delta + \epsilon \cos t)^{-m}z = 0 \tag{1}$$

where m is a positive integer, Δ and ϵ are parameters, and dots represent differentiation with respect to t, is a particular case of Hill's equation. For a given Δ and ϵ, the point (Δ, ϵ) is said to be stable if all solutions of (1) are bounded for all $t > 0$, and unstable if an unbounded solution exists. It is desired to find those regions in the Δ-ϵ plane which are stable.

Panovko and Gubanova [1, pp. 180–193] have shown that equation (1) with $m = 1$ governs the torsional vibrations of a disk attached to one end of an elastic shaft, the other end of which is fixed. If the distance between the fixed end and the disk is varied as $L + A \cos \omega t$, then the equation of motion becomes

$$I\ddot{\theta} + JG(L + A \cos \omega t)^{-1}\theta = 0 \tag{2}$$

where

I = mass moment of inertia of disk
J = polar moment of inertia of shaft
G = shear modulus of shaft
θ = angle of torsional rotation of disk

Taking a new time variable in (2) as ωt, obtain equation (1) with

$m = 1$
$\Delta = I\omega^2 L/JG$
$\epsilon = I\omega^2 A/JG$

The same authors have demonstrated that equation (1) with $m = 3$ governs the bending vibrations of a light elastic cantilevered beam with a concentrated mass at its free end. If the length of the cantilever is varied as $L + A \cos \omega t$, then the equation of motion becomes

$$m\ddot{y} + 3EI(L + A \cos \omega t)^{-3}y = 0 \tag{3}$$

where

m = concentrated mass
I = area moment of inertia of beam cross section
E = Young's modulus of beam
y = displacement of mass

Again taking ωt as independent variable in (3), obtain equation (1) with

$m = 3$
$\Delta = (\omega^2 mL^3/3EI)^{1/3}$
$\epsilon = (\omega^2 mA^3/3EI)^{1/3}$

Note that in both examples it is required that $A < L$ in order for the distance between the disk (or mass) and the support to be greater than zero. Hence, in these examples, the region of physical interest in the Δ-ϵ plane is

$$\Delta > \epsilon > 0$$

Panovko and Gubanova [1] investigated the stability of (1) for $m = 1$ and $m = 3$ by expanding the coefficient of z in a power series in ϵ and neglecting terms of $0(\epsilon^2)$. This reduces (1) to Mathieu's equation, the stability of which is well known [2].

Stability Diagram

The stability diagram of (1) (i.e., the Δ-ϵ plane together with its regions of stability) possesses the following properties:

Property 1. The stability diagram is symmetric about the Δ-axis.

Proof: Replacing ϵ by $-\epsilon$ in (1) is equivalent to replacing t by $t - \pi$. But the definition of stability used herein is independent of any particular initial conditions and hence replacing t by $t - \pi$ cannot effect stability.

Property 2. The stability diagram is unstable for $|\Delta| \leq |\epsilon|$.

Proof: Equation (1) possesses a singularity whenever $|\Delta| \leq |\epsilon|$.

Property 3. For m even, the stability diagram is symmetric about the ϵ-axis.

Proof: Replacing Δ by $-\Delta$ is again equivalent to replacing t by $t - \pi$, since $(-\Delta + \epsilon \cos t)^m = (\Delta - \epsilon \cos t)^m$ for m even.

Property 4. For m odd, the negative Δ-axis of the stability diagram is unstable.

Proof: For $\epsilon = 0$, (1) becomes $\ddot{z} + \Delta^{-m}z = 0$, which possesses unbounded solutions if m is odd and $\Delta \leq 0$.

Property 5. Transition curves separating regions of stability from regions of instability intersect the positive Δ-axis of the stability diagram at

$$\Delta = (2/N)^{2/m}, \qquad N = 1, 2, 3 \ldots \tag{4}$$

Proof: It is well known from Floquet theory [3, p. 201] that corresponding to transition values of Δ and ϵ, there must exist at least one periodic solution to (1) of period Ω or 2Ω, where Ω is the least period of $(\Delta + \epsilon \cos t)^{-m}$, i.e., where $\Omega = 2\pi$. Therefore, in order to obtain all transition values of Δ and ϵ, it is sufficient to examine solutions of period $4\pi/N$, all of which have period 4π. (Here, and in what follows, $N = 1, 2, 3 \ldots$)

Now for $\epsilon = 0$ and $\Delta > 0$, the solutions to (1) are of the form $\sin \Delta^{-m/2}t$ and $\cos \Delta^{-m/2}t$, which have period $2\pi\Delta^{m/2}$. Thus, for $\epsilon = 0$, transition points can occur only if

$$2\pi\Delta^{m/2} = 4\pi/N$$

$$\Delta = (2/N)^{2/m}$$

Consider the set S of transition points on the positive Δ-axis defined by (4) for fixed m.

Property 6. $\Delta = 0$ is an accumulation point of S.

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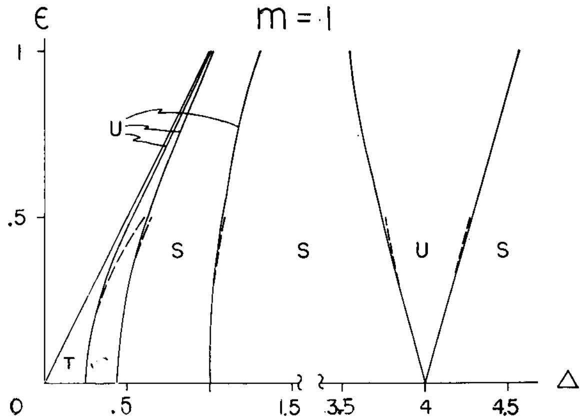

Fig. 1 **Stability diagram for equation (1) with $m = 1$. Computer results are represented by solid lines, perturbation results by dashed lines. U = unstable, S = stable. An infinite number of transition curves lie in region T, accumulating along the line $\Delta = \epsilon$. Points in T may be either stable or unstable. The transition curves for $N = 1, 2, 3, 4$ are shown intersecting the Δ-axis at $\Delta = 4, 1, 0.44, 0.25$, respectively.**

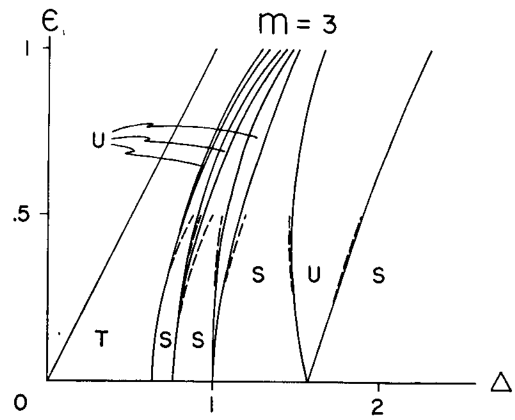

Fig. 2 **Stability diagram for equation (1) with $m = 3$. Computer results are represented by solid lines, perturbation results by dashed lines. U = unstable, S = stable. An infinite number of transition curves lie in region T, accumulating along the line $\Delta = \epsilon$. Points in T may be either stable or unstable. The transition curves for $N = 1, 2, 3, 4$ are shown intersecting the Δ-axis at $\Delta = 1.59, 1, 0.76, 0.63$, respectively.**

Property 7. The largest element of S, $\Delta = 4^{1/m}$, occurs for $N = 1$ and always lies above $\Delta = 1$.

Property 8. $\Delta = 1$ is always an element of S and occurs for $N = 2$.

Property 9. For $N \geq 2$ all the elements of S lie in the interval [0, 1].

Consider now the sequence R of transition points on the positive Δ-axis for a fixed value of N and for $m = 1, 2, 3, \ldots$,

$$\Delta = (2/N)^{2/m}$$

Property 10. $\Delta = 1$ is an accumulation point of R.

See Figs. 1–3.

Perturbation Solution

A perturbation method [3, p. 209] is used to obtain explicit expressions (valid for small ϵ) for the transition curves which intersect the points $\epsilon = 0$, $\Delta = (2/N)^{2/m}$. Expand

$$z(t) = z_0(t) + z_1(t)\epsilon + z_2(t)\epsilon^2 + \ldots \quad (5)$$

$$\Delta = (2/N)^{2/m} + \Delta_1\epsilon + \Delta_2\epsilon^2 + \ldots \quad (6)$$

and substitute (5) and (6) into (1). By equating coefficients of like powers of ϵ, obtain a linear differential equation with constant coefficients on $z_i(t)$. Requiring $z_i(t)$ to be periodic gives a value for Δ_i. z_0 is taken first as $\sin Nt/2$, then as $\cos Nt/2$, since choice gives a separate transition curve.

In this manner the equations for the transition curves to $0(\epsilon^4)$ were obtained

$$N = 1: \quad \Delta = (4)^{1/m} \pm \frac{1}{2}\epsilon + \left(\frac{1}{4}\right)^{1/m}\left(\frac{1}{8} + \frac{5m}{32}\right)\epsilon^2 \mp \left(\frac{1}{4}\right)^{2/m}\left(\frac{59m^2 + 120m + 64}{1536}\right)\epsilon^3 + 0(\epsilon^4) \quad (7)$$

$$N = 2: \quad \Delta = 1 + \left(\frac{3 + 5m}{24}\right)\epsilon^2 + 0(\epsilon^4) \quad (8)$$

$$\Delta = 1 + \left(\frac{9 - m}{24}\right)\epsilon^2 + 0(\epsilon^4) \quad (9)$$

$$N = 3: \quad \Delta = \left(\frac{4}{9}\right)^{1/m} + \left(\frac{9}{4}\right)^{1/m}\left(\frac{16 + 7m}{64}\right)\epsilon^2 \pm \left(\frac{9}{4}\right)^{2/m}\left(\frac{59m^2 - 120m + 64}{1536}\right)\epsilon^3 + 0(\epsilon^4) \quad (10)$$

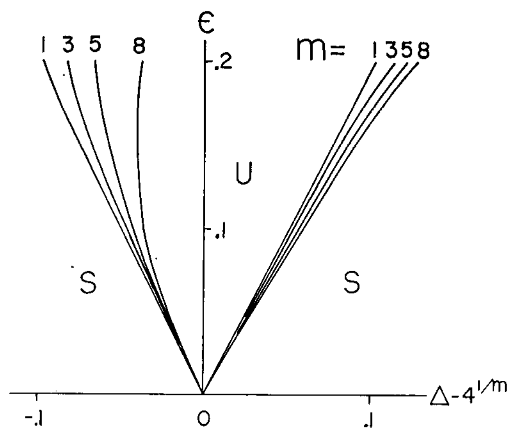

Fig. 3 **Region of instability associated with $N = 1$ for various values of m, equation (7). The Δ-axis is replaced by the $(\Delta\text{-}4^{1/m})$ axis. U = unstable, S = stable.**

$$N \geq 4: \quad \Delta = \left(\frac{4}{N^2}\right)^{1/m} + \left(\frac{N^2}{4}\right)^{1/m}\left(\frac{1}{4}\right)\left[1 + \frac{m}{2}\left(\frac{N^2 - 2}{N^2 - 1}\right)\right]\epsilon^2 + 0(\epsilon^4) \quad (11)$$

It is to be noted that for $m = -1$ these expressions agree with those given by McLachlan [2, p. 16] for Mathieu's equation.

Numerical Solution

Consider two independent solutions of (1), $z^{(1)}(t)$ and $z^{(2)}(t)$, such that

$$\begin{aligned} z^{(1)}(0) &= \dot{z}^{(2)}(0) = 1 \\ \dot{z}^{(1)}(0) &= z^{(2)}(0) = 0 \end{aligned} \quad (12)$$

It follows from Floquet theory [4, p. 134] that if

$$|z^{(1)}(\Omega) + \dot{z}^{(2)}(\Omega)| < 2 \quad (13)$$

then all the solutions of (1) are bounded. If, however, the inequality of (13) is reversed, then there exist unbounded solutions.

This criteria allows a numerical determination of stability. If (1) is written in finite-difference form and $z^{(1)}(t)$ and $z^{(2)}(t)$ are generated from the initial values (12), then examination of $z^{(1)}$ and $\dot{z}^{(2)}$ at $t = \Omega = 2\pi$ determines stability for any given value of m, Δ, and ϵ. A digital computer program to effect this end was written and executed.

The numerical solution is valid for large values of ϵ and is therefore expected to be more accurate than the perturbation solution for points in the Δ-ϵ plane far from the Δ-axis.

Results

Figs. 1 and 2 illustrate the stability diagrams of (1) for $m = 1$ and $m = 3$, respectively. The computer results are shown as solid lines while the perturbation results are dashed. As might be expected from properties 2 and 6, the computer analysis has revealed that $\Delta = \epsilon$ is an accumulation point for the set of transition points lying on the line $\epsilon =$ constant. The region of the stability diagram just to the right of the line $\Delta = \epsilon$ is therefore filled with transition curves. The stability of any particular point in this region must be determined by examining its position relative to the transition curves in its neighborhood. Because of graphical difficulties, only the transition curves for $N = 1, 2, 3, 4$ are shown in Figs. 1 and 2; all transition curves for higher N lie between the $N = 4$ curve and the line $\Delta = \epsilon$. There is good agreement between the perturbation results and the computer results for small ϵ.

Fig. 3 shows the region of instability associated with $N = 1$ for various values of m. The Δ-axis has been replaced by the $(\Delta - 4^{1/m})$ axis and hence all the $N = 1$ transition curves intersect this axis at the origin; see equation (7). Note that the slopes of the $N = 1$ transition curves are independent of m at the Δ-axis.

References

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