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A HIGHER ORDER APPROXIMATION FOR NON-LINEAR NORMAL MODES IN TWO DEGREE OF FREEDOM SYSTEMS

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Abstract—Explicit approximate expressions for normal modal curves in non-linear two degree of freedom systems are obtained by applying a perturbation method to the modal equation:

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0.$$

The results are not valid in the neighborhood of $V = h$.

EXPLICIT approximate expressions for normal modal curves in non-linear two degree of freedom systems have been obtained by Rosenberg and Kuo [1] and by Rand [2]. This paper extends these results to a higher order approximation. The existence of such normal modes has been treated by Pak and Rosenberg [3].

Consider a holonomic, scleronomous conservative system with generalized coordinates x, y. Consider a class of systems for which the potential energy V is of the form

$$V = ax^2 + by^2 + \alpha x^4 + \beta x^3 y + \gamma x^2 y^2 + \sigma x y^3 + \tau y^4, \tag{1}$$

where a, b, α, β, γ, σ, τ are constants such that V is positive definite, and for which the kinetic energy T is of the form

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2), \tag{2}$$

where dots represent differentiation with respect to time t.
Then

$$\ddot{x} = -V_x \tag{3}$$

$$\ddot{y} = -V_y \tag{4}$$

$$T + V = h, \tag{5}$$

where h is a constant equal to the total energy of the system.

Using

$$\dot{y} = y'\dot{x}$$

and

$$\ddot{y} = y''\dot{x}^2 + y'\ddot{x}$$

(2)–(5) give ([4], [5])

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0, \tag{6}$$

where primes denote differentiation with respect to x. Normal modes of the system will be solutions of (2) which satisfy ([5], p. 162, 172)

$$y(x = 0) = 0 \tag{7}$$

and which intersect the curve $V = h$.

A perturbation method will be used to solve (6) approximately.
Set

$$x = h^{\frac{1}{2}}\xi \tag{8}$$

$$y(x) = h^{\frac{1}{2}}\eta(\xi) \tag{9}$$

$$\eta(\xi) = \sum_{m=1}^{\infty} h^m \eta_m(\xi), \tag{10}$$

where the $\eta_m(\xi)$ are as yet unknown. Substitute (8)–(10) into (1) and (6) and collect like powers of h. Equating to zero the coefficient of h^m provides an infinite set of equations for the $\eta_m(\xi)$. The first two of these are

$$(1 - a\xi^2)\frac{d^2\eta_1}{d\xi^2} - a\xi\frac{d\eta_1}{d\xi} + b\eta_1 = -\frac{\beta}{2}\xi^3 \tag{11}$$

$$(1 - a\xi^2)\frac{d^2\eta_2}{d\xi^2} - a\xi\frac{d\eta_2}{d\xi} + b\eta_2 = \alpha\xi^4\frac{d^2\eta_1}{d\xi^2} + 2\alpha\xi^3\frac{d\eta_1}{d\xi} - \gamma\xi^2\eta_1. \tag{12}$$

The complimentary solution to both (11) and (12) is

$$\eta_{m,\text{comp}}(\xi) = C_{1m}\sin(\Delta \arcsin a^{\frac{1}{2}}\xi) + C_{2m}\cos(\Delta \arcsin a^{\frac{1}{2}}\xi),$$

where C_{1m} and C_{2m} are arbitrary constants and where $\Delta = (b/a)^{\frac{1}{2}}$. A particular solution to (11) is

$$\eta_{1,\text{part}}(\xi) = A\xi + B\xi^3,$$

where $A = 6B/(a - b)$

$B = (\beta/2)/(9a - b)$.

Since $\eta_m = \eta_{m,\text{comp}} + \eta_{m,\text{part}}$
then from (7), $C_{21} = 0$. For $d\eta_1/d\xi$ to be bounded at $\xi = a^{\frac{1}{2}}$, $C_{11} = 0$.

Thus $\eta_1(\xi) = \eta_{1,\text{part}}(\xi)$.
A particular solution to (12) is

$$\eta_{2,\text{part}}(\xi) = L\xi + M\xi^3 + N\xi^5,$$

where $L = 6M/(a - b)$

$M = (20N - E)/(9a - b)$

$N = -F/(25a - b)$

$E = A(2\alpha - \gamma)$

$F = B(12\alpha - \gamma)$.

As with $\eta_1(\xi)$, $C_{22} = C_{12} = 0$ and $\eta_2(\xi) = \eta_{2,\text{part}}(\xi)$.

From (8)–(10),

$$y(x) = h^{\frac{3}{2}}\eta_1 + h^{\frac{5}{2}}\eta_2 + 0(h^{\frac{7}{2}})$$

$$y(x) = Ahx + Bx^3 + Lh^2x + Mhx^3 + Nx^5 + 0(h^{\frac{7}{2}}), \tag{13}$$

where A, B, L, M, N are given above.

(13) represents an approximation to the non-linear normal mode corresponding to the linear normal mode $y \equiv 0$. To find an approximation to the non-linear normal mode corresponding to $x \equiv 0$, interchange x and y throughout. The approximation is not valid in the neighborhood of $V = h$ since (6) is singular at $V = h$. The first two terms of (13) agree with the results derived by a different method in [2], valid to $0(h^{\frac{5}{2}})$.

Substituting (13) into (3) gives a single approximate non-linear ordinary differential equation for $x(t)$ for which approximate methods of solution are well known (see, e.g. Minorsky [6]).

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Résumé—On obtient des expressions approximatives explicites des courbes des modes fondamentaux de systèmes non linéaires à deux degrés de liberté en appliquant une méthode de perturbation à partir de l'équation de mode:

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0.$$

Les résultats ne sont pas valables au voisinage de $V = h$.

Zusammenfassung—Explizite Näherungsausdrücke für Normal-„Modalkurven" in nichtlinearen Systemen mit zwei Freiheitsgraden werden durch Anwendung eines Störungsverfahrens auf die folgende „Modal-Gleichung" erhalten:

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0$$

Die Ergebnisse besitzen keine Gültigkeit in der Nähe von $V = h$.

Аннотация—Путем применения пертурбационного метода к модальному уравнению:

$$2(h - V)y'' + (1 + y'^2)(V_y - y'V_x) = 0$$

получаются явные приближенные выражения модальных графиков в нелинейных системах с двумя степенями свободы.

Результаты недействительны в окрестности $V = h$.