On the Stability of Hill's Equation With Four Independent Parameters

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The stability of Hill's equation with four independent parameters is studied by using Floquet theory and perturbations. Examples are given which demonstrate how the resulting analysis may be applied to a wide variety of stability problems.

Introduction

MODERN researchers in mechanics, when confronted with a stability problem involving a Hill's equation, usually proceed in one of the following two ways.

1 The Hill's equation is reduced to a Mathieu's equation, an approximation generally valid only to the first order of some small parameter occurring in the problem. Since the stability of Mathieu's equation is well known, [1, p. 17][2] and [2, p. 505], the stability of the corresponding Hill's equation follows immediately.

2 The stability of the Hill's equation is investigated directly by using Floquet theory and any one of the following techniques: (*a*) perturbations, [3, p. 209], (*b*) Fourier analysis, [3, p. 203], (*c*) numerical integration, [4, p. 134]. These methods give results to any order of accuracy desired, but require relatively extensive analysis.

It is the purpose of this paper to provide a method of immediately determining the stability of certain Hill's equations by reduction to an equation of standard form. The latter, called Hill's equation with four independent parameters, is more general than Mathieu's equation (which may be thought of as Hill's equation with two independent parameters). The resulting stability analysis will generally be more accurate than that obtained by method 1 in the foregoing, and will be applicable to a wider range of equations.

In what follows, method 2(*a*) will be used to investigate the stability of Hill's equation with four independent parameters. Because of the perturbation method employed, explicit algebraic expressions will be derived for the desired regions of stability in parameter space (valid for small values of certain parameters).

Klotter and Kotowski [5] have investigated the stability of Hill's equation with three arbitrary parameters by method 2(*b*). Because of the Fourier analysis method employed, their results were numerical and hence did not include explicit algebraic expressions for the regions of stability.

Stability Analysis

The differential equation

$$\ddot{Z} + (A + B\cos t + C\cos 2t + D\cos 3t)Z = 0 \qquad (1)$$

where A, B, C, D are parameters and dots represent differentiation with respect to t, is called Hill's equation with four independent parameters. For a given A, B, C, and D, the point (A, B, C, D) is said to be stable if all solutions of (1) are bounded for all $t > 0$, and unstable if an unbounded solution exists. It is desired to find those regions in the $ABCD$ parameter hyperspace which are stable.

It is well known from Floquet theory, [3, p. 201], that corresponding to transition values of A, B, C, and D from stability to instability, there must exist at least one periodic solution to (1) of period Ω or 2Ω, where Ω is the least period of the coefficient of Z in (1). If B is not identically zero, then $\Omega = 2\pi$. Therefore, in order to obtain all transition values of A, B, C, and D, it is sufficient to examine solutions of periodic $4\pi/N$, all of which have period 4π. (Here and in what follows, $N = 0, 1, 2 \ldots$.)

Now for $B = C = D = 0$ and $A > 0$, the solutions to (1) are of the form $\sin\sqrt{A}t$ and $\cos\sqrt{A}t$, which have period $2\pi/\sqrt{A}$. Thus, for $B = C = D = 0$, transition points can occur only if

$$2\pi/\sqrt{A} = 4\pi/N$$

or

$$A = N^2/4$$

Note that $N = 0$ corresponds to a constant, which is a solution to (1) when $A = B = C = D = 0$, and which may be thought of as a periodic function of period 4π.

For $B = C = D = 0$ and $A \leq 0$, (1) has unbounded solutions, and hence the entire negative A-axis is unstable.

Thus one expects two transition hypersurfaces to intersect each of the foregoing transition points on the A-axis, one behaving like $\sin Nt/2$, the other like $\cos Nt/2$ for $B = C = D = 0$. (Except for $N = 0$, where one expects a single transition hypersurface behaving like a constant for $B = C = D = 0$.)

To obtain explicit expressions for these transition hypersurfaces valid for small values of B, C, and D, a perturbation method is used [3, p. 209].

Expand

$$Z(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} Z_{ijk}(t)B^iC^jD^k \qquad (2)$$

$$A = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{ijk}B^iC^jD^k \qquad (3)$$

where $A_{000} = N^2/4$, and substitute (2) and (3) into (1). By equating the coefficients of like powers of $B^iC^jD^k$, obtain a linear differential equation with constant coefficients on $Z_{ijk}(t)$. Requiring $Z_{ijk}(t)$ to be periodic gives a value for A_{ijk}. For $N > 0$, Z_{000} is taken first as $\sin Nt/2$, then as $\cos Nt/2$, since each choice gives a separate transition hypersurface.

By maintaining the first 20 terms in (2) and (3) (i.e., those terms for which $i + j + k \leq 3$), the equations for the first seven transition hypersurfaces were obtained as:

$N = 0$:

$$A = -(1/2)B^2 - (1/8)C^2 - (1/18)D^2 - (3/8)B^2C - (7/36)BCD + \ldots \qquad (4)$$

$N = 1$:

$$A = (1/4) \pm (1/2)B \pm (1/4)BC \pm (1/12)CD - (1/8)B^2 - (1/6)C^2 - (1/16)D^2 + (1/24)B^2C \pm (5/96)B^2D \pm (1/24)C^2D \mp (1/72)BC^2 \mp (1/1152)BD^2 \mp (1/32)B^3 - (13/144)BCD + \ldots \qquad (5)$$

$N = 2$:

$$A = 1 + (1/2)C + (1/6)BD - (1/12)B^2 - (1/32)C^2 - (1/10)D^2 - (7/288)B^2C - (2/225)CD^2 - (1/512)C^3 + (11/360)BCD + \ldots \qquad (6)$$

$$A = 1 - (1/2)C - (1/6)BD + (5/12)B^2 - (1/32)C^2 - (1/10)D^2 + (\overset{25}{\cancel{121}}/288)B^2C + (2/225)CD^2 + (1/512)C^3 + \underset{-79/360}{\cancel{(43/180)}}BCD + \ldots \qquad (7)$$

$N = 3$:

$$A = (9/4) \pm (1/2)D \mp (1/4)BC + (1/16)B^2 + (1/10)C^2 - (1/72)D^2 - (3/80)B^2C \mp (9/128)B^2D \mp (9/200)C^2D \pm (1/32)B^3 \mp (1/2592)D^3 + (73/720)BCD + \ldots \qquad (8)$$

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[2] Numbers in brackets designate References at end of Note.

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For $C = D = 0$, (1) reduces to Mathieu's equation and (4)–(8) agree with the expressions given by Kauderer [2, p. 505].

Applications

Example 1. Consider the equation

$$\ddot{Z} + (\delta + \epsilon \cos^3 t)Z = 0 \tag{9}$$

which arises in the study of the stability of periodic solutions to the nonlinear equation [3, p. 114]

$$\ddot{x} + \alpha x + \beta x^4 = F \cos \omega t$$

Using the identity

$$\cos^3 t = \frac{3}{4} \cos t + \frac{1}{4} \cos 3t$$

comparison of (9) with (1) gives

$$\begin{aligned} A &= \delta \\ B &= 3\epsilon/4 \\ C &= 0 \\ D &= \epsilon/4 \end{aligned} \tag{10}$$

By substituting (10) into (4)–(8), one immediately obtains the first seven transition curves in the δ-ϵ plane for (9), to $0(\epsilon^4)$; e.g., for $N = 0$,

$$\delta = -(41/144)\epsilon^2 + 0(\epsilon^4)$$

Note that (9) could not be reduced to a Mathieu equation for small ϵ, and hence method 1 in the foregoing is not applicable in this case.

Example 2. Yang and Rosenberg [6] have shown that the stability of the x-mode of the particle in the plane is dependent only upon the stability of the equation

$$\ddot{Z} + [(\delta - \epsilon \cos^2 t)/(1 - \epsilon \cos^2 t)]Z = 0 \tag{11}$$

where $\delta < 1$, $0 < \epsilon < 1$.

Expand the coefficient of Z in a power series of ϵ,

$$(\delta - \epsilon \cos^2 t)(1 + \epsilon \cos^2 t + \epsilon^2 \cos^4 t + \epsilon^3 \cos^6 t + \ldots).$$

Taking $2t$ as a new time variable, comparison with (1) reveals

$$\begin{aligned} 4A &= \delta + (\delta - 1)[(\epsilon/2) + (3\epsilon^2/8) + (5\epsilon^3/16)] + 0(\epsilon^4) \\ 4B &= (\delta - 1)[(\epsilon/2) + (\epsilon^2/2) + (15\epsilon^3/32)] + 0(\epsilon^4) \\ C &= 0(\epsilon^2) \\ D &= 0(\epsilon^3) \end{aligned} \tag{12}$$

The only physically significant transition curve in the δ-ϵ plane corresponds to $N = 0$ for this problem, since $\delta < 1$. In view of (12), (4) becomes

$$A = -(1/2)B^2 + 0(\epsilon^4) \tag{13}$$

Expanding

$$\delta = \delta_0 + \delta_1\epsilon + \delta_2\epsilon^2 + \delta_3\epsilon^3 + 0(\epsilon^4)$$

in (12) and (13) and equating like powers of ϵ gives the $N = 0$ transition curve as

$$\delta = (1/2)\epsilon + (3/32)\epsilon^2 + (3/64)\epsilon^3 + 0(\epsilon^4)$$

which agrees with the results obtained in [7] by method $2(a)$ in the foregoing.

Yang and Rosenberg [6] investigated the stability of (11) by method 1 in the foregoing. Their results were valid to $0(\epsilon^2)$.

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