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Degenerate homoclinic cycles in perturbations of quadratic Hamiltonian systems

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Abstract. This paper discusses limit cycles of quadratic, planar vector fields. It investigates the number of limit cycles that can be created by a small quadratic deformation of a Hamiltonian, quadratic vector field in a neighbourhood of a homoclinic orbit. Two explicit bounds are given: a rigorous bound of 5 and a numerical bound of 2. The bounds are obtained from perturbation calculations of divergence integrals at a homoclinic cycle.

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Hilbert's 16th problem asks for a bound on the number of limit cycles possessed by polynomial vector fields of degree d. The vector field is defined by a system of differential equations

$$\dot{x} = P(x, y)$$
$$\dot{y} = Q(x, y)$$

with P and Q polynomials of degree d. A *limit cycle* γ is a simple closed curve that is the forward (ω) or backward (α) limit set of a trajectory of this system, disjoint from γ. This paper addresses one small aspect of the Hilbert problem for the quadratic vector fields through the use of computer algebra.

Perturbation methods can be used to obtain information about the number of limit cycles that occur in systems which are close to integrable ones. Arnold [1] has even isolated a 'weakened' Hilbert problem which seeks to place bounds on the number of limit cycles which occur in perturbations of Hamiltonian (or more

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generally integrable) systems of a given degree. To avoid confusion from using the term integral in two different ways, we call a function $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^2$, U open, with the property that $\operatorname{grad} f \cdot X = 0$ a *constant of the motion* for the flow generated by X on U. Let X be integrable, and assume that X has a family of closed orbits γ_c, $c \in (c_0, c_1)$. If $X_\varepsilon = X + \varepsilon Y$ is a family of vector fields with X integrable, then we want to locate limit cycles of X_ε, ε small. Here we assume that Y does not depend upon ε. All computations that follow are first order in ε except where stated otherwise. The basic principle we use is the classical observation that if D is a region and Z is a vector field tangent to the boundary of D, then $\int_D \operatorname{div} Z = 0$. A necessary condition for the existence of a family of limit cycles δ_ε of X_ε with $\delta_0 = \gamma_c$ can be easily obtained from this formula. We must have

$$0 = \frac{d}{d\varepsilon}\left(\iint_{\operatorname{int}\delta_\varepsilon} \operatorname{div} X_\varepsilon\right)\Bigg|_{\varepsilon=0} = \iint_{\operatorname{int}\gamma_c} \operatorname{div} Y = \int_{\gamma_c} Y \times \frac{X}{|X|}.$$

Consequently, for small $\varepsilon > 0$, limit cycles of X_ε depending smoothly on ε as $\varepsilon \to 0$, tend to curves γ_c giving zeros of the function

$$I(c) = \iint_{\operatorname{int}\gamma_c} \operatorname{div} Y.$$

Conversely, Andronov *et al* [2] prove that simple zeros of $I(c)$ do give rise to families of limit cycles of X_ε that vary smoothly with ε. When X has an algebraic constant of the motion, then the $I(c)$ are Abelian integrals and the problem of estimating the number of zeros of I involves the periods of Abelian integrals and their dependence upon parameters.

Families of closed orbits for an integrable system either extend to infinity, or they terminate in centres or singular cycles containing equilibrium points. The behaviour of the integrals $I(c)$ as c tends to a singular value of the constant of motion requires special attention in the latter case because $I(c)$ need not extend to an analytic function on a closed interval whose boundary corresponds to a singular value. The simplest case of singular cycles in integrable systems occurs for perturbations of quadratic Hamiltonian systems. The constant of the motion in this case is a cubic polynomial, and non-singular plane cubic curves are elliptic. This implies that the integrals $I(c)$ are classical complete elliptic integrals. For several restricted families of quadratic Hamiltonian systems, the integrals $I(c)$ have been studied [3, 4]. The general case appears to require lengthy algebraic calculations which we are pursuing with the aid of symbolic manipulation programs on computers. The general properties of asymptotic expansions for $I(c)$ at singular values of c have been studied. This paper carries out the calculation of the first few coefficients in this asymptotic expansion for quadratic vector fields. The expressions for these coefficients are elementary functions, but some of them are so large that a computer is required to calculate them. The asymptotic behaviour of the integrals $I(c)$ near a singular cycle containing one non-degenerate saddle is the primary subject of this paper. A singular plane cubic is a rational curve, so that the integral $I(c)$ is an elementary integral in this case. Nonetheless, our calculations of this quantity required days of computation on a Symbolics 3670 computer running MACSYMA.

Consider an integrable system with a homoclinic orbit γ_{c_0} and a family of perturbations $X + \varepsilon Y$. The function $I(c)$ associated with a family of periodic orbits bounded by γ_{c_0} has an asymptotic expansion containing terms of the form $(c-c_0)^i$ and terms of the form $(c-c_0)^i \ln(c-c_0)$. The terms of the form $(c-c_0)^i \ln(c-c_0)$ are determined algebraically from the power series expansion of the vector field at the saddle via a normal form computation. The terms of the form $(c-c_0)^i$ are given by elementary integrals. We compute these expansions for perturbations of quadratic Hamiltonian systems and find numerically that if the first three terms of the expansion vanish, then the perturbation lies in the direction of integrable perturbations of the Hamiltonian system. Using the work of Roussarie [5], this strongly suggests that the upper bound for the number of limit cycles which can appear in perturbations of a quadratic Hamiltonian system in a neighbourhood of a homoclinic orbit is two. We are not able to give a rigorous proof of this statement, however, for two reasons. First, the numerical calculation of the coefficient of $(c-c_0)$ is so lengthy that we have not attempted a detailed numerical study, let alone a rigorous analysis. Second, our analysis is only first order in ε and hence does not extend readily to a neighbourhood of Hamiltonian systems with integrable, but non-Hamiltonian, perturbations. There might exist regions tangent to the integrable perturbations of Hamiltonian systems yielding systems with more limit cycles.

We begin the analysis by performing an affine change of coordinates so that the quadratic Hamiltonian system we study has a saddle point at the origin with stable manifold the y axis, unstable manifold the x axis, and a homoclinic loop in the third quadrant of the plane. The Hamiltonian H of the system is then

$$H = xy + Q(x, y) \tag{1}$$

where $Q(x, y)$ is cubic in x and y. Dilation of the plane and rescaling of time allows us to assume that the coefficients of the quadratic term xy and the cubic term x^3 are both 1. Note that if the coefficient of x^3 is zero, then the x axis is invariant under the flow and there is no homoclinic loop. We write $Q(x, y)$ in factored form:

$$H = xy + \prod_{i=1}^{3} (x + k_i y) \tag{2}$$

where the k_i are either all real, or one real and a complex conjugate pair. See figure 1 for the level curves of H (for typical values of the parameters k_i.) Note that the level curve $H = 0$ goes through the origin and includes the saddle loop. If the k_i are distinct, whether it has a single asymptote at infinity or two depends on whether some of the k_i are complex.

We shall study saddle-loop bifurcations in the system:

$$x' = H_y + \varepsilon f(x, y) \tag{3a}$$

$$y' = -H_x + \varepsilon g(x, y) \tag{3b}$$

where f and g are quadratic in x and y, and where H is given by (2). The key quantity which governs the saddle-loop bifurcation is the integral of the divergence of the vector field taken over the interior of one of the closed curves $H = h$ lying inside the saddle loop for $h > 0$:

$$\varepsilon I(h) = \iint_{H \geqslant h} \varepsilon\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y. \tag{4}$$

To first order in ε, $I(h) + h$ is the Poincaré return map of (3).

It has been shown (Roussarie [5]: the result is generally attributed to Andronov

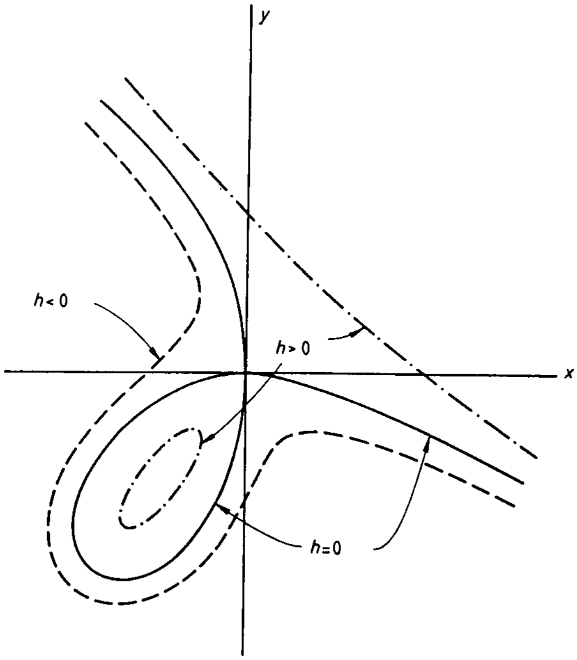

Figure 1. Level curves $H = h$ of the Hamiltonian: equation (2).

and Leontovich) that $I(h)$ has the following asymptotic expansion, valid for small $h > 0$:

$$I(h) = c_1 + c_2 h \ln h + c_3 h + c_4 h^2 \ln h + \dots \qquad (5)$$

in which

$$c_1 = I(0) = \text{the integral (4) taken over the saddle-loop}$$

$$c_2 = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\bigg|_{x=y=0}$$

$$c_3 = \frac{\partial}{\partial h}\,[I(h) - c_2 h \ln h]|_{h=0}$$

and where c_4 will be discussed later. This asymptotic expansion is valid for any homoclinic orbit with a non-degenerate saddle in an integrable system. It can be derived readily by using singularity theory to find coordinates (u, v) for which the constant of motion H is given by uv near the saddle point. Transforming the integral (4) to (u, v) coordinates near the saddle leads immediately to (5).

Our goal is to choose the parameters k_i and the form of the perturbation functions $f(x, y)$, $g(x, y)$ so as to make as many of the constants c_i vanish as possible (in order of ascending i) without making $I(h)$ identically zero. (If $I(h) \equiv 0$ the system will be integrable to first order in ε, and we can draw no conclusions about the number of limit cycles in perturbations.) If we can produce a non-integrable system with $c_1 = c_2 = \dots = c_n = 0 \neq c_{n+1}$, then there will exist perturbations that produce examples with n limit cycles (Roussarie [5]), though it does not follow that there will be quadratic perturbations with n limit cycles.

In order to simplify the computation of the integral c_1, we first consider the condition $c_2 = 0$. This is equivalent to the eigenvalues of the saddle point remaining negatives of one another. By introducing affine coordinate changes and rescalings of time that depend smoothly on the perturbation parameter ε, we may assume that the saddle point and its linearisation remain unchanged. In these coordinates, f and g have no constant or linear terms. Moreover, to produce the most general terms in the integrand for $I(h)$, it suffices to consider

$$f(x, y) = \beta xy \qquad g(x, y) = xy$$

with ε set to the coefficient of g. This yields the system (from (3))

$$x' = H_y + \varepsilon\beta xy \tag{6a}$$

$$y' = -H_x + \varepsilon xy \tag{6b}$$

for which the divergence in (4) becomes

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \beta y + x.$$

The general quadratic vector field with saddle point at the origin, eigenvalues ± 1, stable manifold tangent to the y axis, and unstable manifold tangent to the x axis can be written in the form (6).

In order to calculate the integral c_1, we transform to polar coordinates

$$x = \rho \cos\theta \qquad y = \rho \sin\theta \tag{7}$$

whereupon

$$c_1 = \int_0^{\pi/2} \mathrm{d}\theta \int_0^{r(\theta)} (\cos\theta + \beta \sin\theta)\rho^2 \,\mathrm{d}\rho \tag{8}$$

$$= \frac{1}{3}\int_0^{\pi/2} (\cos\theta + \beta\sin\theta) r(\theta)^3 \,\mathrm{d}\theta \tag{9}$$

in which we obtain $r(\theta)$ from $H = 0$ in (2):

$$r(\theta) = -\sin\theta\cos\theta\left(\prod_{i=1}^{3} (\cos\theta + k_i \sin\theta)\right)^{-1}. \tag{10}$$

To evaluate (9) we set

$$\sin\theta = \frac{2u}{1+u^2} \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad \mathrm{d}\theta = \frac{2}{1+u^2}\,\mathrm{d}u \tag{11}$$

giving

$$c_1 = -\frac{16}{3}\int_0^1 u^3(u^2+1)(u^2-1)^3(u^2-2u\beta-1)\left(\prod_{i=1}^{3}(1+2uk_i-u^2)\right)^{-3} \mathrm{d}u. \tag{12}$$

In order to evaluate (12) in closed form, we factorise the denominator by setting

$$k_i = \frac{1}{2}\left(\frac{1}{s_i} - s_i\right) \tag{13}$$

which gives

$$c_1 = \frac{16}{3}\int_0^1 u^3(u^2+1)(u^2-1)^3(u^2-2u\beta-1)\left[\prod_{i=1}^{3}(u+s_i)\left(u-\frac{1}{s_i}\right)\right]^{-3} \mathrm{d}u. \tag{14}$$

There are two cases to be considered: case (i) in which the s_i are all real, and case (ii) in which $Q(x, y)$ has only one real root, i.e. where s_2 and s_3 are complex conjugates.

In order for (10) to represent a saddle loop as in figure 1, the denominator must not vanish for θ in the first or third quadrants. This requires that those parameters k_i which are real (k_1, k_2, k_3 in case (i), and k_1 in case (ii)) are also positive. In terms of the parameters s_i in (13) and (14), this will be satisfied by requiring $0 < s_i < 1$ if s_i is real.

Furthermore, we scale x and y in (2) so that

$$s_1 = \tfrac{1}{2}. \tag{15}$$

This makes $k_1 = \frac{3}{4}$.

In order to evaluate (14) we used the computer algebra system MACSYMA [6] to expand the integrand in partial fractions and then to obtain the indefinite integral. The result was a large expression (of about 20K characters for case (i), and about 150K characters for case (ii)) which can be written in the form

$$c_1 = c_1(s_2, s_3, \beta) = \beta I_1 + I_2 \tag{16}$$

in which I_1 and I_2 are elementary transcendental functions of s_2 and s_3 only. In case (ii) where s_2 and s_3 are complex conjugates, we set

$$s_{2,3} = R \pm \mathrm{i}M \tag{17}$$

in which case I_1 and I_2 are elementary transcendental functions of R and M only.

For given values of s_2 and s_3, (16) prescribes a value of β such that the resulting system will have $c_1 = c_2 = 0$ (cf (5)). This means that such a system has perturbations that have at least two limit cycles near the homoclinic orbit. Quadratic systems with at least two limit cycles are obtainable from perturbations for which the function $I(h)$ has two simple zeros for $h = 0$. These perturbations are obtained as follows. In equation (3) we take $f(x, y) = \gamma x + \beta xy$ and $g(x, y) = xy$. Begin with $\gamma = 0$ and assume that the Hamiltonian H and β are such that $I(h) = c_1 + c_2 h \ln h + c_3 h + r(h)$ with $c_3 \neq 0$ and $r(h) = o(h)$. Note that c_2 depends only on γ and is non-zero for $\gamma \neq 0$, so we can pick γ so that $0 < c_2/c_3$ is small. Since the homoclinic trajectory is in the third quadrant, $\iint y \neq 0$ and β can be varied so that $0 < c_1/c_2$ is small. If c_1/c_2 and c_2/c_3 are both small and positive, then the functions $c_1 + c_2 h \ln h + c_3 h$ and $I(h)$ each have two zeros close to the origin. The vector field defined by (3) then has two limit cycles close to the homoclinic trajectory of the Hamiltonian system.

In order to examine the possibility of the existence of neighbouring systems with three limit cycles, we set $c_3 = 0$ in (5). We may again evaluate the integral by transforming to polar coordinates, in which case

$$c_3 = \left[\frac{\partial}{\partial h} \int_{\theta_1(h)}^{\theta_2(h)} \mathrm{d}\theta \int_{r_l(\theta,h)}^{r_u(\theta,h)} (\cos\theta + \beta \sin\theta)\rho^2 \, \mathrm{d}\rho \right]_{h=0} \tag{18}$$

with $r_u(\theta, h)$ and $r_l(\theta, h)$ describing branches of the curve $H(h)$ (see figure 2).

Writing

$$g(\theta, h) = \int_{r_l(\theta, h)}^{r_u(\theta, h)} (\cos\theta + \beta \sin\theta)\rho^2 \, \mathrm{d}\rho \tag{19}$$

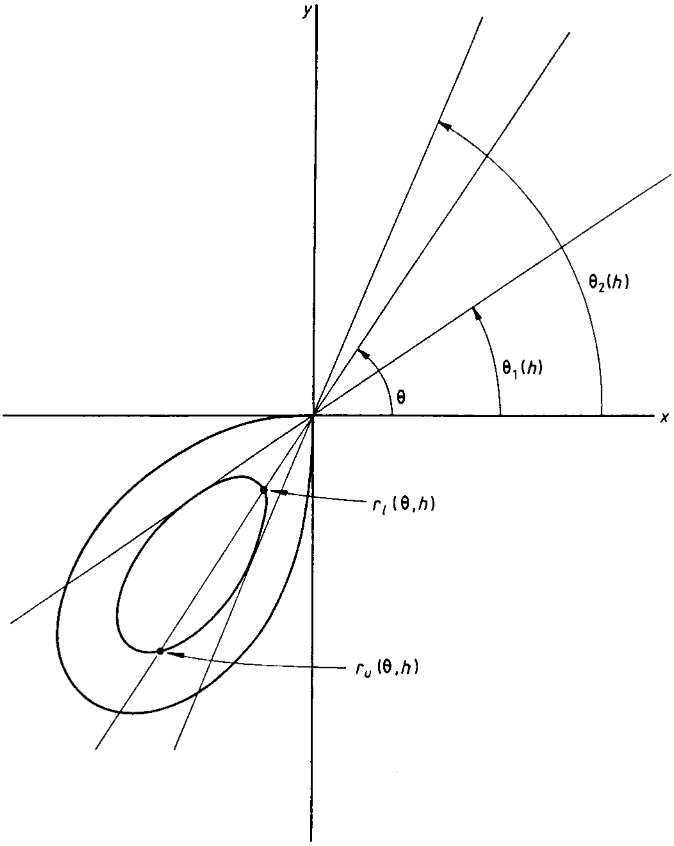

Figure 2. Geometry of the level curves of H in polar coordinates.

(18) becomes

$$c_3 = \int_0^{\pi/2} \frac{\partial g}{\partial h} \mathrm{d}\theta + g\left(\frac{\pi}{2}, 0\right) \frac{\mathrm{d}\theta_2}{\mathrm{d}h}\bigg|_{h=0} + g(0, 0) \frac{\mathrm{d}\theta_1}{\mathrm{d}h}\bigg|_{h=0}. \tag{20}$$

The last two terms in (20) are zero since, e.g., $g(\theta_2(h), h) = 0$, since $r_l(\theta_2(h), h) = r_u(\theta_2(h), h)$; see figure 2. From (19),

$$\frac{\partial g}{\partial h} = \frac{\partial r_u}{\partial h} (\cos\theta + \beta \sin\theta) r_u(\theta, h)^2 - \frac{\partial r_l}{\partial h} (\cos\theta + \beta\sin\theta) r_l(\theta, h)^2. \tag{21}$$

From (2),

$$r_u^2 \cos\theta \sin\theta + r_u^3 Q(\theta) = h \tag{22}$$

where $Q(\theta) = \Pi_{i=1}^3 (\cos\theta + k_i \sin\theta)$, so that

$$\frac{\partial r_u}{\partial h} = \frac{1}{2r_u \cos\theta \sin\theta + 3r_u^2 Q(\theta)} \tag{23}$$

and similarly for r_l. Moreover, when $h = 0$, $r_l = 0$ and r_u is given by (10).

Substitution of these results into (21) and (20) gives

$$c_3 = \int_0^{\pi/2} (\cos\theta + \beta\sin\theta)\left(\prod_{i=1}^{3}(\cos\theta + k_i\sin\theta)\right)^{-1} d\theta. \tag{24}$$

To evaluate (24) we once again make the substitutions (11) giving

$$c_3 = 2\int_0^1 (u^2+1)(u^2-2u\beta-1)\left[\prod_{i=1}^{3}(u+s_i)\left(u-\frac{1}{s_i}\right)\right]^{-1} du. \tag{25}$$

We evaluated (25) using MACSYMA and obtained a closed-form expression:

$$c_3 = c_3(s_2, s_3, \beta) = \beta J_1 + J_2 \tag{26}$$

in which J_1 and J_2 are functions of s_2 and s_3 only (or of R and M in case (ii); see (17)). The full expressions for J_1 and J_2 are given in the appendix.

We solved the equations $c_1 = c_3 = 0$ simultaneously, obtaining from (16) and (26),

$$\beta = \frac{-I_2}{I_1} = \frac{-J_2}{J_1}. \tag{27}$$

The last equation gives a curve in the s_2–s_3 plane (or in the R–M plane in case (ii)) (see figures 3 and 4). We include the following sample points for possible use by the interested reader. In case (i):

$s_2 = 0.21 \qquad s_3 = 0.072\,018\,891\,572\,95\ldots \qquad \beta = -2.275\,952\,339\,172\ldots$

while in case (ii):

$$s_{2,3} = 1 \pm i0.928\,200\,139\,3448\ldots \qquad \beta = -\tfrac{3}{4}.$$

All points on these curves represent systems which either have nearby systems with at most three limit cycles or which are integrable. The computation of these curves was extremely time consuming because of the large size of the expressions for I_i (16). Thus we were unable to demonstrate conclusively that all these systems have integrable perturbations. However, by examining the coefficient c_4, we obtained definitive results.

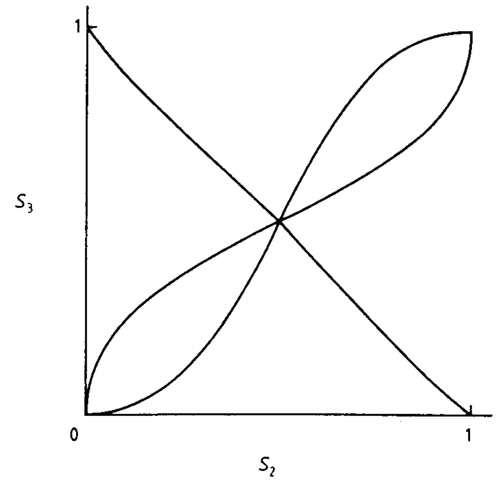

Figure 3. Locus of points in the s_2–s_3 parameter plane (case (i)) for which the integrals c_1 and c_3 simultaneously vanish.

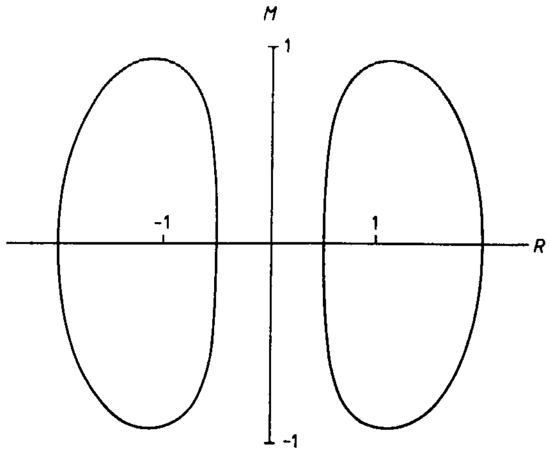

Figure 4. Locus of points in the R–M parameter plane (case (ii)) for which the integrals c_1 and c_3 simultaneously vanish.

The condition $c_4 = 0$ is equivalent to a computation based on the recent work of Joyal and Rosseau [7]. They showed that by using near-identity transformations, any quadratic system with a saddle at the origin

$$x' = x + a_1x^2 + a_2xy + a_3y^2 \tag{28a}$$

$$y' = -y + b_1x^2 + b_2xy + b_3y^2 \tag{28b}$$

can be reduced in a neighbourhood of the origin to the normal form (by analogy with a centre at the origin)

$$u' = u + E_1u^2v + \mathrm{O}(4) \tag{29a}$$

$$v' = -v + E_2uv^2 + \mathrm{O}(4) \tag{29b}$$

where we computed E_1 and E_2 as

$$E_1 = a_2b_2 + \tfrac{2}{3}a_3b_1 - a_1a_2 \tag{30a}$$

$$E_2 = b_2b_3 - \tfrac{2}{3}a_3b_1 - a_2b_2. \tag{30b}$$

The coefficient c_4 in (5) is given by the coefficient of the quadratic term in the divergence of (29). Thus $c_4 = 0$ if and only if

$$2uv(E_1 + E_2) = 0 \tag{31}$$

or

$$b_2b_3 - a_1a_2 = 0. \tag{32}$$

Identifying the a_i, b_i of (28) with our system (6), and substituting into (32) gives the conditions:

$$\beta = \frac{s_1s_3^2s_3^2 + s_1^2s_2s_3^2 + s_1^2s_2^2s_3 - s_2s_3^2 - s_1s_3^2 - s_2^2s_3 - s_1^2s_3 - s_1s_2^2 - s_1^2s_2 + s_1 + s_2 + s_3}{2(s_1s_2s_3^2 + s_1s_2^2s_3 + s_1^2s_2s_3 - s_2s_3 - s_1s_3 - s_1s_2)} \quad \text{case (i)} \tag{33}$$

$$\beta = \frac{R^4 - 3R^3 + 2(M^2-1)R^2 + 3(1-M^2)R + M^4 + 2M^2 + 1}{4R^3 - 3R^2 + 4(M^2-1)R - 3M^2} \quad \text{case (ii).} \tag{34}$$

We then solved (33) (with $s_1 = \frac{1}{2}$) or (34) simultaneously with the condition $c_3 = 0$ (equation (26)). This gave a curve in the s_2–s_3 plane for case (i), or in the R–M plane for case (ii). MACSYMA calculations with numerical precision set to 32 digits indicate that these curves coincide with the previous curves obtained by requiring c_1 and c_3 to simultaneously vanish (figures 3 and 4). This strongly suggests that on the curves in figures 3 and 4 the system is integrable, i.e. that all further coefficients in (5) must vanish once we require c_1, c_2 and c_3 to vanish.

In order to check this hypothesis, we compared the condition for $c_1 = c_3 = c_4 = 0$ with the necessary and sufficient conditions for a quadratic system to have a centre (these are conditions of integrability.) Recall [8] that a system of the form

$$x' = \mu x - \omega y - bx^2 - (2c + B)xy - dy^2 \tag{35}$$

$$y' = \omega x + \mu y + ax^2 + (2b + A)xy + cy^2 \tag{36}$$

will be integrable if and only if $\mu = 0$ and one of the following three conditions is satisfied:

$$\text{(I)}\quad a + c = b + d = 0 \tag{37a}$$

$$\text{(II)}\quad A(a + c) = B(b + d) \quad \text{and} \quad aA^3 - (3b + A)A^2B + (3c + B)AB^2 - dB^3 = 0 \tag{37b}$$

$$\text{(III)}\quad A + 5b + 5d = B + 5a + 5c = ac + bd + 2(a^2 + d^2) = 0. \tag{37c}$$

(The system will be Hamiltonian if and only if $\mu = A = B = 0$.)

In order to apply these results to our system, we must translate the origin to the other equilibrium point (a focus), and then we must perform a linear transformation so that the linear terms of our system are in the canonical form (35), (36).

We wrote a MACSYMA program to accomplish these computations, and then to check the foregoing integrability conditions. We carried terms of order ε^2 and maintained a numerical accuracy of 32 digits. The result was that $\mu = 0$ and condition (II) was satisfied for all points which we checked on the curves in figures 3 and 4.

Joyal and Rousseau [7] and Suillin [9] have calculated c_4, c_6 and c_8 at a saddle point in a manner analogous to the calculation of the 'focal values' that determine whether a singular point is a centre. They prove that if the first three focal values of a quadratic system are zero, then the system is integrable. This implies a rigorous, but larger, bound for the number of limit cycles obtainable by perturbation of a homoclinic orbit.

We have examined their calculations for systems close to Hamiltonian systems and find that, in this case, $c_2 = c_4 = c_6 = 0$ implies that the system is integrable. If the system is expressed in the form

$$x' = x + bx^2 + (2c + \varepsilon B)xy + dy^2 \tag{38a}$$

$$y' = -y - ax^2 - (2b + \varepsilon A)xy - cy^2 \tag{38b}$$

it can be scaled so as to place an equilibrium point at (1, 1) by requiring

$$1 + b + 2c + \varepsilon B + d = 1 + a + 2b + \varepsilon A + c = 0. \tag{39}$$

Then substituting in the Joyal–Rousseau–Suilin expression for c_4 yields

$$c_4 = \varepsilon(Ac - Bb). \tag{40}$$

Solving $c_4 = 0$ for A and substituting into c_0 gives

$$c_6 = \frac{1}{c^3}[\varepsilon B(c-b)(2B\varepsilon + 5c)(B\varepsilon + 2c) \\ \times (bBc\varepsilon + b^2B\varepsilon + c^3 + 3bc^2 + c^2 + 3b^2c + bc + b^3 + b^2)].$$

The only solution of $c_6 = 0$ that is not $O(1/\varepsilon)$ is $b = c$. It is easily seen that if $c_4 = c_6 = 0$ in (38) for small ε, then the system is symmetric with respect to the transformation $(x, y, t) \to (y, x, -t)$. This implies that the system satisfies (37.2) and is integrable. Applying the results of Roussarie [5], we have the following theorem.

Theorem. Let X be a Hamiltonian quadratic vector field that has no non-Hamiltonian integrable perturbations. If γ is a homoclinic orbit of X containing a non-degenerate saddle, then there is a neighbourhood U of γ and a neighbourhood V of X in the space of quadratic vector fields, such that $Y \in V$ has at most *five* limit cycles in U.

Proof. If the asymptotic expansion of $I(h)$ satisfies $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$, then we have proved that the perturbation of X is integrable. If there is an $i \leqslant 6$ with $c_i \neq 0$, then the theorem of Roussarie [30] implies that there are at most five limit cycles of Y in a neighbourhood of γ. □

This bound seems to be far from optimal. The computations described in this paper suggest that the optimal bound on the number of limit cycles is 2. The condition $c_1 = c_2 = c_3 = 0$ has been shown computationally to force the system to be integrable. This conclusion can be expressed precisely in the form of a conjecture whose validity is supported by the numerical evidence we have presented.

Conjecture. Let X be a Hamiltonian quadratic vector field that has no non-Hamiltonian integrable perturbations. If γ is a homoclinic orbit of X containing a non-degenerate saddle, then there is a neighbourhood U of γ and a neighbourhood V of X in the space of quadratic vector fields, such that $Y \in V$ has at most *two* limit cycles in U.

Appendix

In order to give the reader a feeling for the nature of the algebraic expressions involved in this work, we present here the quantities J_1 and J_2 defined in (26), obtained by using the computer algebra system MACSYMA. These expressions are similar to, but much smaller in bulk than, those for I_1 and I_2 of (16). As in (15), s_1 is taken as $\frac{1}{2}$:

$$\begin{aligned} J_1 = -8s_2s_3(&2s_2^2s_3\log(s_3+1) + 3s_2s_3\log(s_3+1) \\ &- 2s_3\log(s_3+1) - 2s_2^2s_3\log(s_3) - 3s_2s_3\log(s_3) \\ &+ 2s_3\log(s_3) - 2s_2\log(s_2+1)s_3^2 + 2s_2\log(s_2)s_3^2 \\ &-2\log(1-s_2)s_2s_3^3 + 2\log(\tfrac{3}{2})s_2s_3^2 + 2s_2^2\log(1-s_3)s_3 \\ &+ 3s_2\log(1-s_3)s_3 - 2\log(1-s_3)s_3 - 3s_2\log(s_2+1)s_3 \end{aligned}$$

$$+ 3s_2 \log(s_2)s_3 - 2\log(\tfrac{3}{2})s_2^2 s_3 - 3\log(1-s_2)s_2 s_3$$
$$+ 2\log(\tfrac{3}{2})s_3 + 2s_2\log(s_2+1) - 2s_2\log(s_2) + 2\log(1-s_2)s_2$$
$$- 2\log(\tfrac{3}{2})s_2/((s_2+2)(2s_2-1)(s_3+2)(s_3-s_2)(2s_3-1)(s_2s_3+1))$$

$$J_2 = -4s_2s_3(2s_2^2s_3^2\log(s_3+1) + 3s_2s_3^2\log(s_3+1)$$
$$- 2s_3^2\log(s_3+1) - 2s_2^2\log(s_3+1) - 3s_2\log(s_3+1)$$
$$+ 2\log(s_3+1) - 2s_2^2s_3^2\log(s_3) - 3s_2s_3^2\log(s_3)$$
$$+ 2s_3^2\log(s_3) + 2s_2^2\log(s_3) + 3s_2\log(s_3) - 2\log(s_3)$$
$$+ 2s_2^2\log(1-s_3)s_3^2 + 3s_2\log(1-s_3)s_3^2 - 2\log(1-s_3)s_3^2$$
$$- 2s_2^2\log(s_2+1)s_3^2 + 2\log(s_2+1)s_3^2 + 2s_2^2\log(s_2)s_3^2$$
$$- 2\log(s_2)s_3^2 - 2\log(1-s_2)s_2^2s_3^2 - 3\log(\tfrac{3}{2})s_2s_3^2$$
$$+2\log(1-s_2)s_3^2 - 3s_2^2\log(s_2+1)s_3 + 3\log(s_2+1)s_3$$
$$+ 3s_2^2\log(s_2)s_3 - 3\log(s_2)s_3 - 3\log(1-s_2)s_2^2s_3$$
$$+3\log(\tfrac{3}{2})s_2^2s_3 + 2\log(1-s_2)s_3 - 3\log(\tfrac{3}{2})s_3$$
$$- 2s_2^2\log(1-s_3) - 3s_2\log(1-s_3) + 2\log(1-s_3)$$
$$+ 2s_2^2\log(s_2+1) - 2\log(s_2+1) - 2s_2^2\log(s_2) + 2\log(s_2)$$
$$+ 2\log(1-s_2)s_2^2 + 3\log(\tfrac{3}{2})s_2 - 2\log(1-s_2))$$
$$/((s_2+2)(2s_2-1)(s_3+2)(s_3-s_2)(2s_3-1)(s_2s_3+1)).$$

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