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THE GEOMETRICAL STABILITY OF NON-LINEAR NORMAL MODES IN TWO DEGREE OF FREEDOM SYSTEMS

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Abstract—The geometrical stability of the non-linear normal mode vibrations of a class of two degree of freedom dynamical systems is studied by utilizing the definitions and analysis of Synge's "Geometry of Dynamics."

It is shown that instabilities can occur for small amplitudes of vibration only if

(a) one of the associated linear normal modes possesses a frequency which is nearly a multiple of the frequency of the other linear normal mode, or

(b) the frequency of one linear normal mode is nearly zero.

NON-LINEAR NORMAL MODES

THE EXISTENCE and properties of normal mode vibrations in non-linear systems have been the subject of much recent research (see, e.g., [1, 2].) Explicit approximate expressions for normal modal curves of a particular class of two degree of freedom systems have been obtained by the author [3]. It is the purpose of this paper to consider the stability of these normal mode vibrations.

Consider a holonomic, scleronomous conservative dynamical system S with generalized coordinates x, y. Consider a class of systems for which the potential energy V is of the form

$$V = ax^2 + by^2 + \alpha x^4 + \beta x^3 y + \gamma x^2 y^2 + \sigma x y^3 + \tau y^4 \tag{1}$$

where a, b, α, β, γ, σ, τ are constants such that V is positive definite, and for which the kinetic energy T is of the form

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{2}$$

and where dots represent differentiation with respect to time t. Then

$$\ddot{x} = -V_x \tag{3}$$

$$\ddot{y} = -V_y \tag{4}$$

$$T + V = h \tag{5}$$

where h is a constant equal to the total energy.

For such a system S, it has been shown [3] that an approximate expression for the non-linear normal mode that corresponds to the linear normal mode $y \equiv 0$ is

$$y = h^{\frac{3}{2}}\eta(\xi) + 0(h^{\frac{5}{2}}) \tag{6}$$

where

$$\eta(\xi) = A\xi + E\xi^3 \tag{7}$$

$$x = h^{\frac{1}{2}}\xi \tag{8}$$

$$A = 6E/(a - b) \tag{9}$$

$$E = (\beta/2)/(9a - b) \tag{10}$$

Substituting (6)–(10) into (3) to find $\xi(t)$ gives

$$\ddot{\xi} + 2a\xi = -4\alpha\xi^3 h + 0(h^2). \tag{11}$$

Suppose at $t = 0$ the system is at the origin. Then.

$$\xi(0) = 0 \tag{12}$$

and $\dot{\xi}(0)$ is found from the two conditions

$$\dot{y}(0) = \dot{x}(0)\frac{dy}{dx}(x = 0) \tag{13}$$

and

$$\dot{x}^2(0) + \dot{y}^2(0) = 2h - 2V(x = 0, y = 0) = 2h \tag{14}$$

Thus

$$\dot{x}(0) = (2h)^{\frac{1}{2}}\left[1 + \left(\frac{dy}{dx}(x = 0)\right)^2\right]^{-\frac{1}{2}} \tag{15}$$

Substituting (6)–(8) into (15), find

$$\dot{\xi}(0) = \sqrt{(2)} + 0(h^2). \tag{16}$$

An approximate solution to equation (11) with the initial conditions (12) and (16) is easily found by perturbations (see, e.g. [4], pp. 98–103.)

$$\xi(t) = a^{-\frac{1}{2}} \sin \sqrt{(2a)}\,\omega t + 0(h) \tag{17}$$

where

$$\omega = 1 + \frac{3\alpha h}{4a^2} + 0(h^2). \tag{18}$$

Equation (17) together with the modal equations (6)–(8) gives the non-linear normal mode $x(t)$, $y(t)$ whose stability is to be considered.

It is to be noted that these expressions are not valid if $b/a = (2M + 1)^2$, $M = 0, 1, 2, \ldots,$ due to vanishing denominators. (The case of $b/a = 1$ has been treated elsewhere [5].) However, these expressions (as well as the ensuing stability analysis) are valid for all other (positive) values of b/a, including values of b/a close to (but not equal to) $(2M + 1)^2$.

GEOMETRICAL STABILITY

Non-linear vibrations are well known to be generally Lyapunov unstable because of the change of their period with a change in amplitude (see, e.g. [4], p. 219.) In order to discuss the stability of non-linear normal modes a definition of stability is needed which gives more physically meaningful results than does Lyapunov stability.

Synge, in a paper of 1926, "On the Geometry of Dynamics" [6], has supplied such a definition of stability. Through the use of tensor calculus he has supplemented his definition with mathematical stability criteria. In what follows, Synge's analysis will be applied to the study of the geometrical stability of non-linear modes. His results are derived for an n-dimensional Riemannian space. These will be specialized to a two dimensional Euclidean space since the metric associated with the kinetic energy (2) is Euclidean:

$$ds^2 = 2T\,dt^2 = dx^2 + dy^2$$

(For a short summary of Synge's paper, see [7], Section 186.)

Consider a motion of the dynamical system S which corresponds to a normal mode. Let C be the curve which this normal mode traces out in the xy plane. Let C^* be the curve corresponding to any other motion of S. Now let 0 be a point of C and 0^* be a point of C^*. O^* and 0 are said to correspond if 0 is at the foot of a perpendicular dropped from 0^* onto C. Let $d(0, 0^*)$ represent the usual Euclidean distance between 0 and 0^*.

Let δ be any positive number and consider all those curves C^* which possess a point 0^* corresponding to some point 0 of C, such that

$$d(0, 0^*) < \delta$$

Every curve C^* satisfying this condition is called a disturbed curve of order δ.

Definition (Synge): The curve C is said to be geometrically stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $d(P, P^*) < \epsilon$ for every pair of corresponding points P and P^*, with P on C, and P^* on any disturbed curve of order δ. (Note: Synge defined three types of geometrical stability. This definition, "the most interesting from a physical point of view," he calls stability in the kinematico-statical sense.)

Thus geometrical stability is orbital stability in configuration space. (Orbital stability is usually defined in phase space. See, e.g., [8], Section 1.8).

Synge, in Chapter 8 of [6], derives for two degree of freedom systems the following linearized ordinary differential equation for $B(t)$, an infinitesimal perturbation from P on the modal curve C to the corresponding point P^* on C^*. (Here t is the time for P to move along C).

$$\ddot{B} + B(v^2K + V_{mn}v^m v^n + 3v^2\kappa^2) + 2\kappa(\delta h) = 0 \tag{19}$$

where

v = velocity of the motion along C
K = Gaussian curvature of the configuration space at C
$V_{mn} = \nabla\nabla V$
v^m = unit vector normal to C
κ = curvature of C
δh = infinitesimal difference in energy between C^* and C and where dots represent differentiation with respect to t.

Synge concludes that in order for a motion along C of the system S to be geometrically stable, it is necessary and sufficient that all solutions to this equation remain bounded for all t, for arbitrary values of the infinitesimal constant δh.

For the normal mode problem considered in this paper the quantities appearing in equation (19) are found as follows. From (2), (5),

$$v^2 = 2(h - V).$$

Since the conflguration space is a Euclidean plane,

$$K = 0.$$

The unit normal to the curve $C: x = x(s)$, $y = y(s)$ is

$$v'^n = (-\mathrm{d}y/\mathrm{d}s, \mathrm{d}x/\mathrm{d}s).$$

Therefore,

$$V_{,mn}v'^m v^n = \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 \frac{\partial^2 V}{\partial x^2} + \left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 \frac{\partial^2 V}{\partial y^2} - 2\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}y}{\mathrm{d}s}\frac{\partial^2 V}{\partial x \partial y}$$

$$= (y'^2 V_{xx} + V_{yy} - 2y' V_{xy})/(1 + y'^2)$$

where $y' = \mathrm{d}y/\mathrm{d}x$. The curvature κ of C is ([9], p. 599)

$$\kappa = (V_x y' - V_y)/[2(h - V)(1 + y'^2)^{\frac{1}{2}}].$$

Thus the equation on $B(t)$ becomes

$$\ddot{B} + (1 + y'^2)^{-1}\left(y'^2 V_{xx} + V_{yy} - 2y' V_{xy} + \frac{3}{2}\frac{[V_x y' - V_y]^2}{h - V}\right)B$$

$$= (V_y - y' V_x)(\delta h)/[(h - V)(1 + y'^2)^{\frac{1}{2}}]. \quad (20)$$

The coefficients in this equation are to be understood as functions of t, with $x(t)$ and $y(t)$ given by the normal mode whose stability is to be considered.

Substituting (6)–(10) into (20) gives, after some algebra,

$$\ddot{B} + (2b + 2\gamma h[\xi(t)]^2 + 0(h^2))\,B = Fh^{\frac{1}{2}}\xi(t) + 0(h^{\frac{3}{2}}). \quad (21)$$

where

$$F = -6\beta(\delta h)/(9a - b)$$

Substituting (17) into (21) gives

$$\ddot{B} + [2b + (2\gamma h/a)\sin^2 \sqrt{(2a)}\,\omega t + 0(h^2)]\,B = F\sqrt{(h/a)}\sin \sqrt{(2a)}\,\omega t + 0(h^{\frac{3}{2}}). \quad (22)$$

Now set $z = 2\sqrt{(2a)}\,\omega t$ and (22) becomes

$$8a\omega^2 B'' + [2b + (\gamma h/a) - (\gamma h/a)\cos z + 0(h^2)]\,B = F\sqrt{(h/a)}\sin(z/2) + 0(h^{\frac{3}{2}}), \quad (23)$$

where primes now denote differentiation with respect to z, and where ω is given by (18).

To $0(h^{\frac{3}{2}})$ this is a nonhomogeneous Mathieu equation. Kotowski [10] has studied this equation and has found that if the circular frequency of the nonhomogeneous term is either an integer or a half integer, then there is no difference in the stability charts of the nonhomogeneous equation and the homogeneous equation. Since the circular frequency of the nonhomogeneity in equation (23) is $\frac{1}{2}$, it is sufficient to consider only the corresponding homogeneous equation. (Only the boundedness of the solutions $B(z)$ is of interest in this discussion. The particular form of these solutions, different for the homogeneous and nonhomogeneous cases, is not of interest here).

The resulting homogeneous Mathieu equation becomes, to $0(\epsilon^2)$,

$$B'' + (\delta + \epsilon \cos z)\,B = 0 \quad (24)$$

where

$$\delta = \frac{b}{4a} + \frac{\gamma h}{8a^2} - \frac{3\alpha b h}{8a^3} \quad (25)$$

$$\epsilon = -\frac{\gamma h}{8a^2}. \tag{26}$$

As is well known ([4], [9], [11]), for small ϵ there exist unbounded solutions to equation (24) only if δ is close to $N^2/4$, $N = 1, 2, 3, \ldots$, or $\delta < 0$. Therefore the non-linear normal mode corresponding to the linear normal mode $y \equiv 0$ can be geometrically unstable for small h only if b/a is close to N^2 or to zero. Note that b/a is the square of the ratio of the frequency of the linear normal mode $x \equiv 0$ to the frequency of the linear normal mode $y \equiv 0$. Thus instability for small h can occur only if (a) the frequency of one linear normal mode is nearly a multiple of the frequency of the other, or (b) the frequency of one linear normal mode is nearly zero. This is reminiscent of the instabilities associated with frequency commensurabilities in celestial mechanics ([12], p. 386).

This phenomena has been observed in non-linear vibrations as autoparametric excitations ([13], p. 506). Physically, the condition that the frequency of one normal mode be nearly a multiple of the other evidently allows energy (which is conserved) to be redistributed between the modes. Although only one mode be present initially, a small perturbation produces a component of the other mode, this component growing in time in the unstable case.

To $O(\epsilon^2)$ the only regions of instability on the δ–ϵ stability chart which are significant are those which intersect the δ axis at $\delta = 0$ and $\delta = \frac{1}{4}$. (The others have no width to $O(\epsilon^2)$. See [4], [9], [11]). The corresponding transition curves (separating regions of stability from regions of instability) are ([4], p. 212)

$$\delta = 0 + O(\epsilon^2) \tag{27}$$

$$\delta = (1/4) \pm (\epsilon/2) + O(\epsilon^2). \tag{28}$$

Substituting (25) and (26) into (27) and (28) gives the following transition curves in the $b/a - \epsilon$ plane

$$b/a = 4\epsilon + O(\epsilon^2) \tag{29}$$

$$b/a = 1 + [4 \pm 2 - 12(\alpha/\gamma)]\,\epsilon + O(\epsilon^2). \tag{30}$$

These transition curves are shown in Fig. 1.

For similar results relating to the geometrical stability of the non-linear normal mode corresponding to the linear normal mode $x \equiv 0$, interchange x and y throughout.

EXAMPLE

Yang and Rosenberg [14], [15] have studied a dynamical system called the particle in the plane, for which

$$T = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

$$V = (1/4)(d_1 - L)^2 + (1/4)(d_2 - L)^2$$

where

$$d_{1,2}^2 = (1 \pm x)^2 + y^2$$

and where

$$L > 0.$$

This system possesses a similar (straight line) non-linear normal mode $y \equiv 0$ (the x-mode). Its stability has been previously studied [14]–[18].

Expanding V in a Taylor series,

$$V = (x^2/2) + [(1 - L)\, y^2/2] - (Lx^2y^2/2) + (Ly^4/8) + \text{higher order terms} \quad (31)$$

Comparing (31) with (1), find

$$a = \tfrac{1}{2}$$
$$b = (1 - L)/2$$
$$\alpha = \beta = \sigma = 0$$
$$\gamma = -L/2$$
$$\tau = L/8.$$

Therefore $b/a = 1 - L$. Since $L > 0$, $b/a < 1$. Since $\gamma = -L/2 < 0$, $\epsilon > 0$. From Fig. 1, the

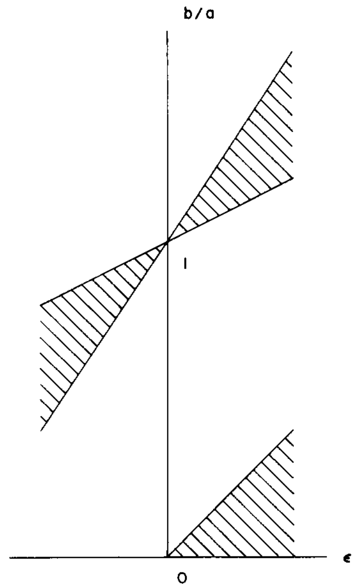

FIG. 1. Geometrical stability of a non-linear normal mode. Transition curves (equations (29), (30)) separating regions of stability (unshaded) from regions of instability (shaded). ϵ is given by equations (26). In equation (30), α has peen taken equal to zero.

only region of instability which lies in the region $0 < b/a < 1$, $\epsilon > 0$, occurs for b/a near zero. From equation (29), the condition for stability becomes, to $0(\epsilon^2)$,

$$b/a > 4\epsilon = -\gamma h/(2a^2)$$

or

$$h < (1 - L)/L \quad (32)$$

The amplitude X of the x-mode is, from (17) and (8),

$$X = (h/a)^{\frac{1}{2}} = \sqrt{(2h)}. \tag{33}$$

From equations (32) and (33), the condition for stability becomes

$$X^2 < 2(1 - L)/L. \tag{34}$$

Equation (34) agrees to $0(\epsilon^2)$ with the results of previous stability studies (see e.g. [14], p. 18). This is an example of instability occurring when the frequency of one linear normal mode (the y-mode) is nearly zero.

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Zusammenfassung—Die geometrische Stabilität der nichtlinearen Eigenschwingungen einer Klasse dynamischer Systeme mit zwei Freiheitsgraden wird an Hand der Definitionen und der Berechnung in "Geometry of Dynamics" von Synge untersucht.

Es wird gezeigt, dass für kleine Schwingungsamplituden Instabilitäten nur dann auftreten können, wenn

(a) eine der zugehörigen linearen Eigenschwingungen eine Frequenz besitzt, die nahezu ein einzahliges Vielfaches der Frequenz der anderen linearen Eigenschwingung ist, oder wenn

(b) die Frequenz einer linearen Eigenschwingung nahezu gleich null ist.

Аннотация—Исследована геометрическая устойчивость нормальных мод нелинейных колебаний класса динамических систем с двумя степенями свободы. Исследование основано на анализе и определениях, данных в книге Синга «Геометрия динамики».