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A Perturbation Solution for the Stability of the Triangular Points in the Elliptic Restricted Problem[1]

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Abstract

A general perturbation technique is used to study the stability of infinitesimal motions about the triangular libration points in the elliptic restricted problem of three bodies. The slopes of the transition curves bounding regions of stability in the $\mu - e$ plane are found in closed form for small e and $\mu \approx 0.02859$.

Introduction

Danby [1] and Bennett [2] have used Floquet theory in a numerical analysis of the stability of infinitesimal motions about the triangular equilibrium points in the elliptic restricted problem of three bodies. Their results are shown in Fig. 1, where

e = eccentricity of the orbit of the primaries

μ = ratio of the mass of the smaller primary to the sum of the masses of the primaries

$$\mu_a = \tfrac{1}{2}[1 - (23/27)^{1/2}] = 0.03852 \cdots$$

$$\mu_b = \tfrac{1}{2}[1 - (24/27)^{1/2}] = 0.02859 \cdots .$$

In this paper a method developed by Cochran [3] for treating singular perturbation problems is used to obtain an algebraic expression for the slopes of the transition curves at $\mu = \mu_b$. As expected from the numerical work of Danby [1] and Bennett [2] the two slopes at $\mu = \mu_b$ differ only by sign, i.e., there is local symmetry about $\mu = \mu_b$ in the $\mu - e$ plane.

The definition of stability used here is the same as that used by Danby [1], p. 165: "a point (which may be moving) is considered to be stable only if an arbitrary but infinitesimal displacement from the point is followed by motion that forever remains infinitesimally close to the point."

First Variational Equations

The equations of infinitesimal motions of the third body about the triangular libration points as derived by Szebehely [4], p. 598, are

$$u'' - 2v' = g(e, f)[\Omega^0_{xx}u + \Omega^0_{xy}v] \quad (1a)$$

$$v'' + 2u' = g(e, f)[\Omega^0_{xy}u + \Omega^0_{yy}v] \quad (1b)$$

where u, v are the pulsating dimensionless coordinates of the third body relative to the triangular point L_4, f is the true anomaly of the smaller primary

$$g(e, f) = 1/(1 + e\cos f) \quad (2)$$

$$\Omega^0_{xx} = 3/4 \quad (3)$$

$$\Omega^0_{xy} = 3\sqrt{3}(\mu - \tfrac{1}{2})/2 \quad (4)$$

$$\Omega^0_{yy} = 9/4 \quad (5)$$

and where primes denote differentiation with respect to f.

A principal system of coordinates may be derived, after Szebehely [4], p. 254, by setting

$$u = x\cos\alpha - y\sin\alpha \quad (6a)$$

$$v = x\sin\alpha + y\cos\alpha \quad (6b)$$

where

$$\tan 2\alpha = \sqrt{3}(1 - 2\mu). \quad (7)$$

Then the equations of motion become

$$x'' - 2y' - g(e, f)f_2 x = 0 \quad (8a)$$

$$y'' + 2x' - g(e, f)f_1 y = 0 \quad (8b)$$

where

$$f_{1,2} = \tfrac{3}{2}\{1 \pm [1 - 3\mu(1 - \mu)]^{1/2}\}. \quad (9)$$

Perturbation Scheme

Elementary Considerations

x, y and g are now expanded in a power series of the eccentricity e:

$$x(f) = \sum_{n=0}^{\infty} x_n(f)\, e^n = x_0(f) + e x_1(f) + \cdots \quad (10a)$$

$$y(f) = \sum_{n=0}^{\infty} y_n(f) e^n = y_0(f) + e y_1(f) + \cdots \quad (10b)$$

$$g(e, f) = (1 + e\cos f)^{-1}$$
$$= 1 - e\cos f + e^2\cos^2 f - \cdots . \quad (11)$$

Substituting Eqs. (10) and (11) into Eq. (8) and

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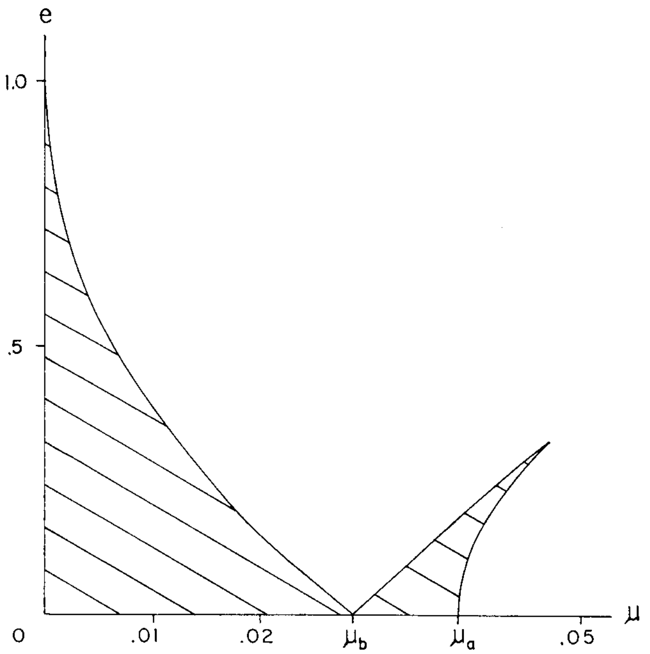

FIG. 1. Transition curves for infinitesimal motions about the triangular equilibrium points in the elliptic restricted problem of three bodies. Shaded regions represent stability. Danby [1].

equating the coefficients of like powers of e to zero, gives the following equations:

1st Order (e^0)

$$x_0'' - 2y_0' - f_2 x_0 = 0 \tag{12a}$$

$$y_0'' + 2x_0' - f_1 y_0 = 0 \tag{12b}$$

2nd Order (e^1)

$$x_1'' - 2y_1' - f_2 x_1 = -f_2 x_0 \cos f \tag{13a}$$

$$y_1'' + 2x_1' - f_1 y_1 = -f_1 y_0 \cos f. \tag{13b}$$

The first-order equations are just the equations of motion for the circular problem and the solution is (see, for instance, Szebehely [4], p. 257).

$$x_0 = A_1 \cos \lambda_1 f + B_1 \sin \lambda_1 f + A_2 \cos \lambda_2 f + B_2 \sin \lambda_2 f \tag{14a}$$

$$y_0 = \alpha_1 B_1 \cos \lambda_1 f - \alpha_1 A_1 \sin \lambda_1 f + \alpha_2 B_2 \cos \lambda_2 f - \alpha_2 A_2 \sin \lambda_2 f \tag{14b}$$

where

$$\alpha_i = \frac{2\lambda_i}{\lambda_i^2 + f_1} = \frac{\lambda_i^2 + f_2}{2\lambda_i}. \tag{15}$$

The frequencies λ_1 and λ_2 are the roots of

$$\lambda^4 - \lambda^2(4 - f_1 - f_2) + f_1 f_2 = 0. \tag{16}$$

For $\mu < \mu_a$, λ_1 and λ_2 are real, and therfore x_0 and y_0 represent stable solutions.

Substitution of the first-order solution, Eqs. (14), into the second-order Eqs. (13) gives

$$\begin{aligned} x_1'' - 2y_1' - f_2 x_1 = -\frac{f_2}{2}\{ & A_1[\cos(1+\lambda_1)f + \cos(1-\lambda_1)f] + B_1[\sin(1+\lambda_1)f \\ & - \sin(1-\lambda_1)f] + A_2[\cos(1+\lambda_2)f + \cos(1-\lambda_2)f] + B_2[\sin(1+\lambda_2)f \\ & - \sin(1-\lambda_2)f]\} \end{aligned} \tag{17a}$$

$$\begin{aligned} y_1'' + 2x_1' - f_1 y_1 = -\frac{f_1}{2}\{ & \alpha_1 B_1[\cos(1+\lambda_1)f + \cos(1-\lambda_1)f] - \alpha_1 A_1[\sin(1+\lambda_1)f \\ & - \sin(1-\lambda_1)f] + \alpha_2 B_2[\cos(1+\lambda_2)f + \cos(1-\lambda_2)f] - \alpha_2 A_2[\sin(1+\lambda_2)f \\ & - \sin(1-\lambda_2)f]\}. \end{aligned} \tag{17b}$$

Inspection of Eqs. (17) shows that no secular terms will appear in x_1 or y_1 except when $\lambda_1 = \frac{1}{2}$, i.e., when $\mu = \mu_b$. This phenomena was observed numerically by Danby [1], p. 166.

Therefore, to $0(e^2)$ in x and y, instability cannot occur for $\mu < \mu_a$ unless $\mu = \mu_b$. The rest of this paper will be concerned with the stability of infinitesimal motions about the triangular points for $\mu \approx \mu_b$.

Stability Analysis for $\mu \approx \mu_b$:

Let f be replaced by two new independent variables ξ and η,

$$\xi = f \tag{18}$$

$$\eta = \omega f = \sum_{n=1}^{\infty} \omega_n e^n f = e\omega_1 f + e^2 \omega_2 f + \cdots, \tag{19}$$

where ξ is of order one for small f, η is of order one for large f, and the constants ω_n are as yet undetermined.

Since x and y are now functions of the independent variables ξ and η, their derivatives become

$$\frac{d}{df} = \frac{\partial}{\partial \xi} + \omega \frac{\partial}{\partial \eta} \tag{20}$$

$$\frac{d^2}{df^2} = \frac{\partial^2}{\partial \xi^2} + 2\omega \frac{\partial^2}{\partial \xi \, \partial \eta} + \omega^2 \frac{\partial^2}{\partial \eta^2}. \tag{21}$$

Expanding the mass ratio μ in a power series of e about $\mu = \mu_b$ gives

$$\mu = \mu_b + e\mu_1 + e^2 \mu_2 + \cdots. \tag{22}$$

Since f_1 and f_2 are functions of μ, their expansions become

$$\begin{aligned} f_1 &= \sum_{n=0}^{\infty} a_n e^n = a_0 + e a_1 + \cdots \\ f_2 &= \sum_{n=0}^{\infty} b_n e^n = b_0 + e b_1 + \cdots \end{aligned} \tag{23}$$

where

$$a_0 = \tfrac{3}{2}\{1 + [1 - 3\mu_b(1 - \mu_b)]^{1/2}\}$$
$$b_0 = \tfrac{3}{2}\{1 - [1 - 3\mu_b(1 - \mu_b)]^{1/2}\} \tag{24}$$

and
$$b_1 = -a_1 = \frac{9}{4}\frac{(1 - 2\mu_b)\mu_1}{[1 - 3\mu_b(1 - \mu_b)]^{1/2}}. \tag{25}$$

Substituting Eqs. (10), (11), (20), (21) and (23) into the equations of motion (8) yields:

1st order (e^0)

$$x_{0\xi\xi} - 2y_{0\xi} - b_0x_0 = 0 \tag{26a}$$

$$y_{0\xi\xi} + 2x_{0\xi} - a_0y_0 = 0 \tag{26b}$$

2nd order (e^1)

$$x_{1\xi\xi} - 2y_{1\xi} - b_0x_1 = -2\omega_1x_{0\xi\eta} + 2\omega_1y_{0\eta} + b_1x_0 - b_0x_0 \cos \xi \tag{27a}$$

$$y_{1\xi\xi} + 2x_{1\xi} - a_0y_1 = -2\omega_1y_{0\xi\eta} - 2\omega_1x_{0\eta} + a_1y_0 - a_0y_0 \cos \xi \tag{27b}$$

where the subscripts ξ and η denote partial differentiation.

The solution of the first-order equations is

$$x_0 = A_1(\eta) \cos \lambda_1\xi + B_1(\eta) \sin \lambda_1\xi + A_2(\eta) \cos \lambda_2\xi + B_2(\eta) \sin \lambda_2\xi \tag{28a}$$

$$y_0 = \alpha_1B_1(\eta) \cos \lambda_1\xi - \alpha_1A_1(\eta) \sin \lambda_1\xi + \alpha_2B_2(\eta) \cos \lambda_2\xi - \alpha_2A_2(\eta) \sin \lambda_2\xi \tag{28b}$$

where

$$\alpha_i = \frac{2\lambda_i}{\lambda_i^2 + a_0} = \frac{\lambda_i^2 + b_0}{2\lambda_i} \tag{29}$$

and the λ_i are the roots of

$$(\lambda_i^2 + a_0)(\lambda_i^2 + b_0) - 4\lambda_i^2 = 0, \tag{30}$$

i.e., $\lambda_1 = \tfrac{1}{2}$, $\lambda_2 = \sqrt{3}/2$.

This solution is of the same form as Eqs. (14), except here the A_i and B_i are functions of η, since Eqs. (26) are partial differential equations.

Substituting the first-order solution, Eqs. (28), into the second-order equations (27), and using $1 - \lambda_1 = \lambda_1$, yields

$$\begin{aligned} x_{1\xi\xi} - 2y_{1\xi} - b_0 x_1 &= C_1 \cos \lambda_1 \xi + D_1 \sin \lambda_1 \xi \\ &+ C_2 \cos \lambda_2 \xi + D_2 \sin \lambda_2 \xi \\ &- \frac{b_0}{2} A_1 \cos (1 + \lambda_1)\xi - \frac{b_0}{2} B_1 \sin (1 + \lambda_1)\xi \\ &- \frac{b_0}{2} A_2 \cos (1 + \lambda_2)\xi - \frac{b_0}{2} B_2 \sin (1 + \lambda_2)\xi \\ &- \frac{b_0}{2} A_2 \cos (1 - \lambda_2)\xi + \frac{b_0}{2} B_2 \sin (1 - \lambda_2)\xi \end{aligned} \tag{31a}$$

$$\begin{aligned} y_{1\xi\xi} + 2x_{1\xi} - a_0 y_1 &= E_1 \cos \lambda_1 \xi + F_1 \sin \lambda_1 \xi \\ &+ E_2 \cos \lambda_2\xi + F_2 \sin \lambda_2 \xi \\ &- \frac{a_0}{2} \alpha_1 B_1 \cos (1 + \lambda_1)\xi + \frac{a_0}{2} \alpha_1 A_1 \sin (1 + \lambda_1)\xi \\ &- \frac{a_0}{2} \alpha_2 B_2 \cos (1 + \lambda_2)\xi + \frac{a_0}{2} \alpha_2 A_2 \sin (1 + \lambda_2)\xi \\ &- \frac{a_0}{2} \alpha_2 B_2 \cos (1 - \lambda_2)\xi - \frac{a_0}{2} \alpha_2 A_2 \sin (1 - \lambda_2)\xi \end{aligned} \tag{31b}$$

where

$$\begin{aligned} C_1 &= 2\omega_1(\alpha_1 - \lambda_1)B_{1\eta} + \left(b_1 - \frac{b_0}{2}\right) A_1 \\ D_1 &= -2\omega_1(\alpha_1 - \lambda_1)A_{1\eta} + \left(b_1 + \frac{b_0}{2}\right) B_1 \\ C_2 &= 2\omega_1(\alpha_2 - \lambda_2)B_{2\eta} + b_1 A_2 \\ D_2 &= -2\omega_1(\alpha_2 - \lambda_2)A_{2\eta} + b_1 B_2 \\ E_1 &= 2\omega_1(\alpha_1 \lambda_1 - 1)A_{1\eta} + \alpha_1 \left(a_1 - \frac{a_0}{2}\right) B_1 \\ F_1 &= 2\omega_1(\alpha_1 \lambda_1 - 1)B_{1\eta} - \alpha_1 \left(a_1 + \frac{a_0}{2}\right) A_1 \\ E_2 &= 2\omega_1(\alpha_2 \lambda_2 - 1)A_{2\eta} + \alpha_2 a_1 B_2 \\ F_2 &= 2\omega_1(\alpha_2 \lambda_2 - 1)B_{2\eta} - \alpha_2 a_1 A_2. \end{aligned} \tag{32}$$

The solution of the second-order equations (31) is

$$\begin{aligned} x_1 &= [(\lambda_1^2 + a_0)D_1 + 2\lambda_1E_1]\beta_1\xi \cos \lambda_1\xi \\ &+ [-(\lambda_1^2 + a_0)C_1 + 2\lambda_1F_1]\beta_1\xi \sin \lambda_1\xi \\ &+ [(\lambda_2^2 + a_0)D_2 + 2\lambda_2E_2]\beta_2\xi \cos \lambda_2\xi \\ &+ [-(\lambda_2^2 + a_0)C_2 + 2\lambda_2F_2]\beta_2\xi \sin \lambda_2\xi \\ &+ \text{non-secular terms} \end{aligned} \tag{33a}$$

$$\begin{aligned} y_1 &= [-2\lambda_1C_1 + (\lambda_1^2 + b_0)F_1]\beta_1\xi \cos \lambda_1\xi \\ &- [2\lambda_1D_1 + (\lambda_1^2 + b_0)E_1]\beta_1\xi \sin \lambda_1\xi \\ &+ [-2\lambda_2C_2 + (\lambda_2^2 + b_0)F_2]\beta_2\xi \cos \lambda_2\xi \\ &- [2\lambda_2D_2 + (\lambda_2^2 + b_0)E_2]\beta_2\xi \sin \lambda_2\xi \\ &+ \text{non-secular terms} \end{aligned} \tag{33b}$$

where

$$\beta_i = \frac{1}{2\lambda_i(1 - 2\lambda_i^2)}. \tag{34}$$

For Eqs. (33) to be uniformly valid, the secular terms must be eliminated, i.e., the coefficients of $\xi \cos \lambda_1\xi$,

$\xi \sin \lambda_1\xi$, $\xi \cos \lambda_2\xi$, and $\xi \sin \lambda_2\xi$ must vanish. This condition gives

$$\begin{aligned} (\lambda_i^2 + a_0)D_1 + 2\lambda_i E_i &= 0 \\ 2\lambda_i D_i + (\lambda_i^2 + b_0)E_i &= 0 \end{aligned} \qquad i = 1, 2 \qquad (35)$$

and

$$\begin{aligned} -(\lambda_i^2 + a_0)C_i + 2\lambda_i F_i &= 0 \\ -2\lambda_i C_i + (\lambda_i^2 + b_0)F_i &= 0. \end{aligned} \qquad i = 1, 2 \qquad (36)$$

These are four sets of two linear homogeneous algebraic equations, each set in two of the unknowns C_i, D_i, E_i and F_i. For a nontrivial solution the determinant of the coefficients of each set must vanish. This is indeed the case, since the determinants of the coefficients of Eqs. (35) and (36) are just the characteristic equation (30). Therefore,

$$D_i = -\frac{(\lambda_i^2 + b_0)}{2\lambda_i} E_i = -\alpha_i E_i, \qquad i = 1, 2 \qquad (37)$$

$$C_i = \frac{\lambda_i^2 + b_0}{2\lambda_i} F_i = \alpha_i F_i, \qquad i = 1, 2. \qquad (38)$$

Consider the elimination of the λ_2 secular terms first. Substitution of Eqs. (32) into (37) and (38) for $i = 2$ gives

$$\begin{aligned} 2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)B_{2\eta} + (b_1 + \alpha_2^2 a_1)A_2 &= 0 \\ -2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)A_{2\eta} + (b_1 + \alpha_2^2 a_1)B_2 &= 0. \end{aligned} \qquad (39)$$

Assuming solutions of the form $e^{p\eta}$, the characteristic equation of the system (39) is

$$p^2 + \left[\frac{b_1 + \alpha_2^2 a_1}{2\omega_1(2\alpha_2 - \lambda_2 - \alpha_2^2\lambda_2)}\right]^2 = 0. \qquad (40)$$

Since the roots of this equation are imaginary, A_2 and B_2 are oscillatory terms and no instability occurs.

Now consider the elimination of the λ_1 secular terms. Substitution of Eqs. (32) into (37) and (38) for $i = 1$ yields:

$$\begin{aligned} &2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)B_{1\eta} \\ &\qquad + \left[b_1 - \frac{b_0}{2} + \alpha_1^2\left(a_1 + \frac{a_0}{2}\right)\right]A_1 = 0 \\ &-2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)A_{1\eta} \\ &\qquad + \left[b_1 + \frac{b_0}{2} + \alpha_1^2\left(a_1 - \frac{a_0}{2}\right)\right]B_1 = 0. \end{aligned} \qquad (41)$$

Assuming solutions of the form $e^{s\eta}$, the characteristic equation of the system (41) is

$$s^2 + \frac{\left[b_1(1 - \alpha_1^2) + \frac{1}{2}(b_0 - \alpha_1^2 a_0)\right] \cdot \left[b_1(1 - \alpha_1^2) - \frac{1}{2}(b_0 - \alpha_1^2 a_0)\right]}{[2\omega_1(2\alpha_1 - \lambda_1 - \alpha_1^2\lambda_1)]^2} = 0 \qquad (42)$$

where $a_1 = -b_1$ has been used. For A_1 and B_1 to be oscillatory, the roots of this equation must be imaginary. This condition gives

$$b_1^2 > \frac{1}{4}\frac{(b_0 - \alpha_1^2 a_0)^2}{(1 - \alpha_1^2)^2}. \qquad (43)$$

Substituting for a_0, b_0 and b_1 gives

$$\mu_1^2 > \mu^{*2} \qquad (44)$$

or

$$\pm\mu_1 > \mu^* \qquad (45)$$

where

$$\mu^* = \frac{\kappa}{3(1 - 2\mu_b)}\left[\frac{\kappa(1 + \alpha_1^2)}{(1 - \alpha_1^2)} - 1\right] \qquad (46)$$

$$\kappa = [1 - 3\mu_b(1 - \mu_b)]^{1/2}. \qquad (47)$$

Therefore the equations of the transition curves become

$$\mu = \mu_b \pm e\mu^* + O(e^2) \qquad (48)$$

and the slopes at $\mu = \mu_b$ are

$$\frac{de}{d\mu} = \pm\frac{1}{\mu^*} = \pm 24(6/11)^{1/2} = \pm\ 17.72\ldots. \qquad (49)$$

Thus the two slopes differ only by sign and the transition curves exhibit local symmetry about $\mu = \mu_b$. The value 17.72 ... is in agreement with the numerical results of Danby [1] and Bennett [2]. (See Fig. 1.)

Points on the transition curves, Eq. (48), do not satisfy the inequality (43) and thus, to $O(e^2)$, represent unstable infinitesimal motions about the triangular points.

Summary

The stability of infinitesimal motions about the triangular libration points in the elliptic restricted problem of three bodies has been investigated using a perturbation method. In this method the independent variable f was replaced by two independent variables, one of $O(1)$ for large f, the other of $O(1)$ for small f. The first-order solution was found to be of the same form as the solution to the circular restricted problem ($e = 0$), except that the coefficients of integration were functions of that independent variable which is of $O(1)$ for large f. The second-order equation was then investigated and the condition that the secular terms vanish produced a differential equation for the coefficients of integration. For stability these coefficients had to be bounded functions. This condition gave a first approximation for the transition curves.

Higher order approximations can be obtained by continuing the perturbation procedure. (See, for instance, Davis and Alfriend [5], where this perturbation procedure is used to study Van der Pol's equation.)

References

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