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A DIRECT METHOD FOR NON-LINEAR NORMAL MODES

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Abstract—This paper presents a direct method for locating normal modes in certain holonomic, scleronomous, conservative non-linear two degree of freedom dynamical systems. The method does not require that the system studied be close to a linear system.

INTRODUCTION

The existence, properties, and approximation of non-linear normal modes have been the subject of much recent research (see, for example, [1–3]).

This paper presents a direct method for locating normal modes in certain holonomic, scleronomous, conservative non-linear two degree of freedom dynamical systems. Here "direct" means that the application of the method does not explicitly involve the differential equations of motion. This method, an extension of Whittaker's (1902) criterion for the discovery of periodic orbits[4], does not require that the system studied be close to a linear system.

NON-LINEAR NORMAL MODES

Consider a two degree of freedom dynamical system with generalized coordinates x, y, for which the kinetic energy T has the form

$$T = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2)$$

and for which the potential energy $V(x, y)$ has the following properties [1, p. 160]:

(a) $V(-x, -y) = V(x, y)$
(b) The partial derivatives V_x, V_y vanish only at the origin, $V_x(0,0) = V_y(0,0) = 0$
(c) $V(x, y)$ is positive definite.

The origin 0 is the only equilibrium configuration of this generally non-linear system. Any trajectory of this system which passes through 0 must, because of property (a), be symmetric with respect to 0 in the xy plane [1, p. 171].

From the conservation of energy,

$$T+V = h$$

where h is a constant equal to the total energy of the system. Since $T \geqslant 0$, the motion must remain in that region of the xy plane defined by

$$V(x, y) \leqslant h$$

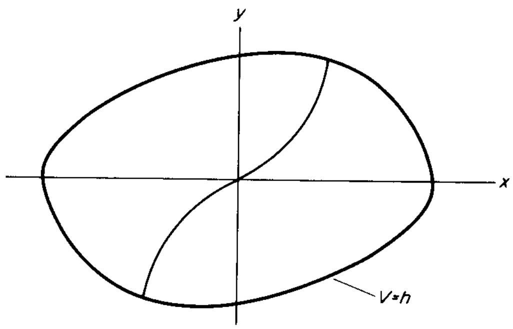

Fig. 1. A non-linear normal mode.

Because of properties (b) and (c), the curves $V(x, y) = \text{constant}$ form a continuum of simple, closed, nonintersecting curves surrounding the origin [1, p. 179].

A normal mode is a periodic motion of this dynamical system which passes through the origin 0 and which has two rest points (located symmetrically with respect to 0) on the curve $V(x, y) = h$ [1, p. 173]. A normal modal curve will further be required to not intersect itself. See Fig. 1.

A DIRECT METHOD

Consider within the region $V(x, y) \leqslant h$ of the xy plane, the set S of all straight line segments which pass through the origin, each reaching from one side of the curve $V = h$ to the other. Any two of these straight line segments divide the region $V \leqslant h$ into two parts, each part consisting of all those straight line segments in S lying between the two given segments. See Fig. 2.

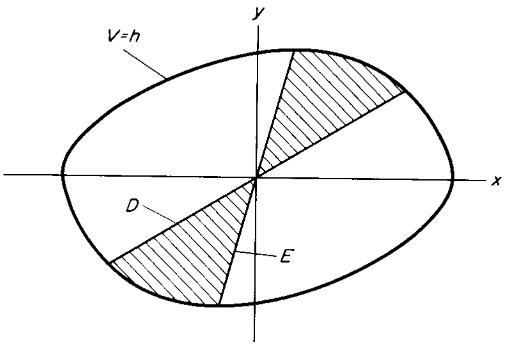

Fig. 2. Two straight line segments which pass through the origin divide the region $V \leqslant h$ into two parts, one shown shaded, the other unshaded.

Now consider the function $G(x, y) = xV_y - yV_x$ [1, p. 184]. By property (a) above, $G(-x, -y) = G(x, y)$. By property (b), $G(0, 0) = 0$.

Theorem

If $G(x, y) = xV_y - yV_x$ is positive at each point (except the origin) of a straight line segment D in the set S, and if $G(x, y)$ is negative at each point (except the origin) of another straight line segment E in S, then there exists a normal mode in the region generated by rotating D counterclockwise into E, shown shaded in Fig. 2.

Proof. (After Whittaker[4].) The curves in the xy plane corresponding to trajectories of this dynamical system are governed by Jacobi's form of the principle of least action: The curve which the system actually follows in traveling between two fixed points gives the action integral

$$\int_1^2 [h - V(x, y)]^{\frac{1}{2}}\, \mathrm{d}s, \qquad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

a stationary value as compared with all other curves joining these two fixed points.

Let D be a straight line segment through the origin, reaching from one side of the curve $V(x, y) = h$ to the other. D can be represented parametrically in terms of s, the distance along D from the origin:

$$D: \quad \begin{aligned} x(s) &= s\cos a \\ y(s) &= s\sin a \end{aligned} \qquad (-L \leqslant s \leqslant L)$$

Here a is the angle between D and the x axis, and $2L$ is the length of D.

Let D' be a smooth curve very close to D with the following properties: D' is symmetric with respect to the origin; D' intersects D only at the origin; D' does not intersect itself; D' has its endpoints on the curve $V = h$. D' can also be represented parametrically in terms of s:

$$D': \quad \begin{aligned} \mathbf{x}(s) &= s\cos a - eh(s)\sin a \\ \mathbf{y}(s) &= s\sin a + eh(s)\cos a \end{aligned} \qquad (-L \leqslant s \leqslant L)$$

Here e is a small positive number and $h(s)$ is the normal distance from D to D'. We shall require that $h(0) = 0$ and $h(s) \neq 0$ for $0 < |s| \leqslant L$ in order that D' intersect D only at the origin. Moreover, $h(-s) = -h(s)$ for D' to be symmetric with respect to the origin.

Let I_D be the value of the action integral taken along the straight line D,

$$I_D = \int_{-L}^{L} \{h - V[x(s), y(s)]\}^{\frac{1}{2}}\, \mathrm{d}s$$

and let $I_{D'}$ be the corresponding quantity for D',

$$I_{D'} = \int_{-L}^{L} \{h - V[\mathbf{x}(s), \mathbf{y}(s)]\}^{\frac{1}{2}} \frac{\mathrm{d}s_1}{\mathrm{d}s}\, \mathrm{d}s$$

where $\mathrm{d}s_1^2 = \mathrm{d}\mathbf{x}^2 + \mathrm{d}\mathbf{y}^2 = \mathrm{d}s^2(1 + e^2 h'(s)^2)$.

Now $I_{D'}$ is a function of e. Expanding $I_{D'}$ in a Taylor series about $e = 0$,

$$\begin{aligned} I_{D'}(e) &= I_{D'}(0) + e\frac{\mathrm{d}I_{D'}}{\mathrm{d}e}(0) + 0(e^2) \\ &= I_D + \delta I_D + 0(e^2) \end{aligned}$$

where

$$\begin{aligned} \delta I_D &= e\frac{\mathrm{d}I_{D'}}{\mathrm{d}e}(0) \\ &= \frac{e}{2}\int_{-L}^{L} \frac{y(s)V_x[x(s), y(s)] - x(s)V_y[x(s), y(s)]}{\{h - V[x(s), y(s)]\}^{\frac{1}{2}}} \frac{h(s)}{s}\, \mathrm{d}s \end{aligned}$$

[The vanishing of $(h - V)^{\frac{1}{2}}$ in the denominator at $s = \pm L$ does not cause δI_D to be infinite. See Appendix.]

Thus if $G(x, y) = xV_y - yV_x$ is positive at each point of D (except 0) then δI_D will be negative for curves D' corresponding to $h(s)/s > 0$. Denote as the positive (negative) side of D that region of the xy plane near D which is generated when D is rotated counterclockwise (clockwise) slightly. Then the value of the action integral will be decreased when any curve D' lying on the positive side of D is taken as the path of integration.

Let E be another such straight line segment through the origin. If $G(x, y)$ is negative at each point of E (except 0) then the value of the action integral will be decreased when any curve E' lying on the negative side of E is taken as the path of integration.

Now the lines D and E divide the region $V(x, y) \leqslant h$ into two parts (Fig. 2). Consider the set T of all smooth non-self-intersecting curves which pass through the origin 0, which are symmetric with respect to 0, which intersect the curve $V = h$, and which lie entirely in the shaded region of Fig. 2. The curve K in T which gives the smallest value of the action integral cannot be D or E, and cannot coincide with D or E for any part of its length (since the curves D' and E' in this region give smaller values of the action integral than do D and E respectively). It follows that the curve K gives a stationary value of the action integral as compared with all curves in T adjacent to K. The curve K is therefore a normal mode. Q.E.D.

EXAMPLE

Consider the system shown in Fig. 3. Two particles of unit mass are constrained to a straight line and restrained by three springs as shown. Let F be the tension in such a spring and let d be its elongation beyond its unstretched length. Then it is assumed that for the

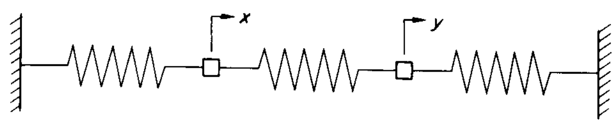

Fig. 3. A two degree of freedom dynamical system.

left, center, and right springs respectively, $F = 20d$, $F = 4d^3$, and $F = 2d$. If x and y, the displacements from equilibrium, are taken as generalized coordinates, the potential energy V becomes

$$V(x, y) = 10x^2 + y^2 + (x-y)^4$$

The function $G(x, y) = xV_y - yV_x$ becomes

$$G(x, y) = -18xy - 4(x+y)(x-y)^3$$

Inspection of the function G reveals that on $x = 0$, $G = 4y^4 > 0$ (except at 0), while on $y = x$, $G = -18x^2 < 0$ (except at 0). Therefore by the above theorem there exists a normal mode in the region obtained by rotating $x = 0$ counterclockwise into $y = x$.

A stronger result follows by noting that on $y = -x$, $G = 18x^2 > 0$ (except at 0), while on $y = 0$, $G = -4x^4 < 0$ (except at 0). There is therefore a normal mode in the region obtained by rotating the line $y = -x$ counterclockwise into the line $y = 0$. Moreover, this result holds for all values of h.

Still stronger results may be obtained for particular values of h by finding appropriate straight line segments along which G has constant sign within the region $V \leqslant h$ (rather than in the entire plane as above). It is generally useful for this purpose to plot the curves $G = 0$ and $V = h$.

These results have been verified by numerical integration of the differential equations of motion.

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APPENDIX

It was shown above that

$$\delta I_D = \frac{e}{2}\int_{-L}^{L} \frac{y(s)V_x[x(s), y(s)] - x(s)V_y[x(s), y(s)]}{\{h - V[x(s), y(s)]\}^{\frac{1}{2}}} \frac{h(s)}{s} \, ds$$

It will now be shown that the vanishing of $(h - V)^{\frac{1}{2}}$ in the denominator at $s = \pm L$ does not cause δL_D to be infinite.

Since

$$\frac{d}{ds}(h - V)^{\frac{1}{2}} = -\tfrac{1}{2}(h - V)^{-\frac{1}{2}}\left(V_x \frac{dx}{ds} + V_y \frac{dy}{ds}\right)$$

and since along the straight line segment D, $dx/ds = x/s$ and $dy/ds = y/s$,

$$\delta I_D = -e\int_{-L}^{L} F(s)h(s)\frac{d}{ds}(h - V)^{\frac{1}{2}} \, ds$$

where

$$F(s) = \frac{yV_x - xV_y}{xV_x + yV_y}$$

Note that $F(\pm L)$ will remain bounded unless the line D is tangent to the curve $V(x, y) = h$. It will be assumed that this unusual case does not occur.

Integrating by parts,

$$\delta I_D = -e(h - V)^{\frac{1}{2}}F(s)h(s)\Big|_{-L}^{L} + e\int_{-L}^{L} (h - V)^{\frac{1}{2}}\frac{d}{ds}[F(s)h(s)] \, ds$$

The first term on the right hand side of this equation vanishes since $h - V = 0$ at $s = \pm L$. The kernel of the remaining integral is obviously bounded at $s = \pm L$, and therefore δI_D is bounded.

Résumé—Cet article présente une méthode directe pour situer les modes normaux de certains systèmes dynamiques, holonomes, scléronomes, conservatifs, non linéaires à deux degrés de liberté. La méthode n'impose pas que le système étudié soit proche d'un système linéaire.

Zusammenfassung—Diese Arbeit beschreibt eine direkte Methode zur Auffindung der Eigenschwingungen in bestimmten holonomischen, skleronomischen, konservativen, dynamischen Systemen mit zwei Freiheitgraden. Es ist nicht erforderlich, dass das untersuchte System nahezu linear ist.