A Thin-Shell Model of the Cornea and Its Application to Corneal Surgery

Howard C. Howland, PhD, Richard H. Rand, PhD, Sharon R. Lubkin, MS

ABSTRACT

Background: We present a thin-shell, analytical model of the vertebrate cornea to study the changes in shape resulting from surgical operations (eg, radial keratotomy).

Methods: A simple closed-form solution is derived for a thin linearly elastic spherical model of the cornea. We assume that the shell is symmetrical about a central axis and that the modulus of elasticity is the same in all directions. The surgery is modeled by allowing the modulus of elasticity (or equivalently the thickness) of the shell to depend upon position, measured as an angle from the axis of symmetry.

Results: The analytical nature of the solution allows us to compute the principal curvatures of the cornea explicitly. For example, for representative parameters, the model predicts the average corneal curvature changes from about 43 diopters before keratotomy to about 38 D after keratotomy.

Conclusions: The model is used to estimate Young's modulus from experimental data reported previously by Thomas et al (*Invest Ophthalmol Vis Sci* 1991;32:1000), as well as to investigate the effect of surgery on corneal flattening and the associated sensitivity to intraocular pressure changes. (*Refract Corneal Surg* 1992;8:183-186.)

From the Section of Neurobiology and Behavior (Dr Howland), Department of Theoretical and Applied Mechanics (Dr Rand), and Center for Applied Mathematics (Ms Lubkin), Cornell University, Ithaca, NY.

Partially supported by NIH Grant EY-02994 to Dr Howland. An abstract of this work was presented at the 1990 Annual Meeting of the Association for Research in Vision and Ophthalmology.[11] *A more technical discussion of portions of this work has been previously published in the* Journal of Biomechanical Engineering.[8]

The authors have no proprietary interest in any research or materials presented within this article.

Reprint requests should be addressed to Howard C. Howland, PhD, Section of Neurobiology and Behavior, Seeley G. Mudd Hall, Cornell University, Ithaca, NY 14853.

Received: July 18, 1991
Accepted: September 20, 1991

A problem with refractive surgery is a relative lack of predictability of the refractive state after the surgery.[1] Recently, finite difference methods[2] and finite element methods[3-6] have been employed in an effort to rationalize the methods for predicting the mechanical and hence refractive effects of corneal surgery. In contrast to these numerical treatments, Mow[7] presented an analytical solution for the cornea treated as a complete sphere with point symmetry. This permitted the stresses in the corneal wall to be expressed as a function of radial position, r. However, the results of this model could not be readily used to study corneal surgery, which by necessity requires that the shell properties be permitted to vary from point to point.

MATERIALS AND METHODS

The cornea is modeled as a thin elastic spherical shell with axisymmetry (ie, symmetry about a central axis) and isotropy in the shell surface (ie, the modulus of elasticity is the same in all directions tangent to the shell surface). It is assumed to be loaded by a uniform hydrostatic pressure representing the effect of the aqueous humor in the anterior chamber (ie, the intraocular pressure). The surgery is modeled by permitting Young's modulus, E, and shell thickness, h, to depend upon the colatitude, ϕ .

The model, which is described in detail in a previous publication,[8] may be represented by the following equation, which relates the radial displacement, w, to the other parameters of the model (Fig 1):

$$w = \frac{a^2 p(1-\nu)}{2\, B(\phi)} \tag{1}$$

Using equation 1, we shall be able to calculate the curvatures of the corneal surface resulting from corneal surgery, the latter modeled by assuming functions E (ϕ) and h (ϕ) in equation 1.

To model radial keratotomy in a mathematically simple way, we average the effects of these incisions around the symmetry axis of the shell. This approxi-

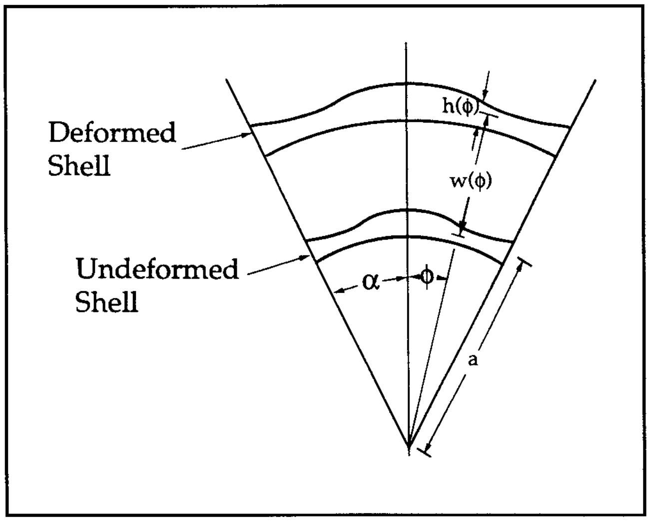

Figure 1: Shell geometry. The cornea is modeled as a thin shell having symmetry about a central axis (shown vertical). The angle ϕ, called the colatitude, determines the position of a circle of points, all of which share the same radial displacement $w(\phi)$. The thickness of the shell at position ϕ is denoted by $h(\phi)$. The undeformed shell radius is a and the value of ϕ at the edge of the cornea is α.

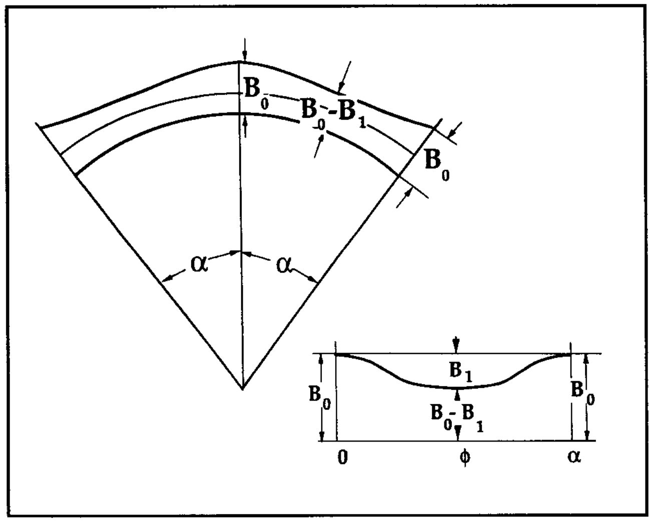

Figure 2: Model of radial keratotomy. To model the effects of corneal surgery, we choose the colatitude ϕ-dependence of the stiffness $B(\phi) = E(\phi)\ h(\phi)$ to reflect the effective local structural weakening due to the surgical cuts, in accordance with equation 2. Thus, after surgery, B remains unchanged ($=B_0$) at $\phi = 0$ (the pole) and $\phi = \alpha$ (the edge of the shell), while it decreases to B_0-B_1 at the point of greatest weakening, which is modeled as occurring at $\phi = \alpha/2$. Thus, B_1/B_0 represents the extent of the surgical cut (eg, if $B_1/B_0 = 0.6$, then the smallest value of B after surgery is 40% of its original (uncut) value.

mation is expected to work best away from the immediate neighborhood of the surgery, for example in the optic zone.

To model the effects of corneal surgery, we choose the colatitude ϕ-dependence of the stiffness $B(\phi) = E(\phi)\ h(\phi)$ to reflect the effective local structural weakening due to the surgical cuts. This includes the effects of changes in both Young's modulus E as well as in shell thickness h. It would be desirable to be able to relate the function $B(\phi)$ directly to the geometry of the surgical cuts, but such a relation is unavailable at the present time. In lieu of such a relation, we choose to write (cf[3]) (in the case of radial keratotomy):

$$B(\phi) = B_0 - \frac{1}{2} B_1 \left[1 - \cos \frac{2\pi}{\alpha} \phi\right] \tag{2}$$

where B_0 = stiffness before surgery, assumed independent of ϕ, and

B_1 = modulus of structural weakening.

Thus, after surgery, B remains unchanged ($=B_0$) at $\phi = 0$ (the pole) and $\phi = \alpha$ (the edge of the shell), while it decreases to B_0-B_1 at the point of greatest weakening, where $\phi = \alpha/2$. Thus, B_1/B_0 represents the extent of the surgical cut (eg, if $B_1/B_0 = 0.6$, then the smallest value of B after surgery is 40% of its original [uncut] value [Fig 2]).

When equation 2 is substituted into equation 1, we obtain an analytical expression for the predicted shape of the deformed corneal surface. The average curvature κ_{av} of this surface is important in determining its optical properties. The average curvature is given by:[8]

$$\kappa_{av} = -\frac{1}{a} + \frac{p(1-\nu)}{2B_0} + \frac{p(1-\nu)}{B_0}\left[\frac{\pi}{\alpha}\right]^2 \frac{B_1}{B^0}\left[1+\frac{2}{3}C\phi^2 + \ldots\right] \tag{3}$$

where

$$C = \frac{1}{2} - 2\left[\frac{\pi}{\alpha}\right]^2 + 6\left[\frac{\pi}{\alpha}\right]^2 \frac{B_1}{B_0} \tag{4}$$

To evaluate this curvature, we used the values of the physical parameters given in Table 1.

RESULTS

For the rest of this article, we take the value of Young's modulus E_0 to be 5×10^6 dynes/cm^2 which falls in the middle of the range of values given in the literature. We find that for the undeformed spherical shell ($p = 0$) $\kappa_{av} = 1/a$, where a is the shell radius. (For convenience, we will drop the minus sign on κ_{av} in what follows. For a = 7.7 mm the BASIC program gives κ_{av} = 43.83 D [Table 2]). (For convenience, we write corneal curvature in diopters [=D], multiplying curvature in mm^{-1} by 337.5, a factor based on the difference between the refractive indices of cornea and air, which expresses the effective power of the

Table 1
Parameter Values

Variable Name	Symbol	Value
Corneal radius[12 (p 489)]	a	7.7 mm
Intraocular pressure[13 (p 408)]	p	2×10^4 dynes/cm^2
Poisson's ratio	υ	0.5
Corneal angle[13 (p 12)]	α	$\pi/3$ radians
Young's modulus[14,15]	E_0	$10^6 - 3 \times 10^7$ dynes/cm^2
Corneal thickness[13 (p 12)]	h_0	0.6 mm
Corneal stiffness	B_0	E_0h_0
Modulus of weakening	B_1	0.5 B_0

Table 2
BASIC Program*

```
10 CLS
20 PRINT "output on "; TIME$, DATE$
30 REM here are some parameters
40 A=7.7 :REM mm radius of cornea
50 NU=.5
60 P = 20000:REM dynes/cm^2 intraocular pressure
70 PI=4*ATN(1)
80 AL=PI/3
90 EO=5E +06:REM dynes/cm^2 modulus of elasticity
100 HO=.6:REM mm thickness of the cornea
110 BO=EO*HO
120 INPUT "FRACTIONAL THICKNESS OF CUT";FTC
130 B1=FTC*BO:'FRACTIONAL THICKNESS OF THE
    CUT.
140 C=1/2-2*(PI/AL)^2+6*(PI/AL)^2*(B1/BO)
150 INPUT "PHI (SUGGEST A NUMBER BETWEEN 0 AND
    0.2 TO STAY IN OPTIC ZONE)"; PHI
160 K=-1/A+P*(1-NU)/(2*BO)+(P*(1-NU)/BO)*(PI/
    AL)^2*(B1/BO)*(1+2*C*PHI^2/3)
170 PRINT
180 PRINT "AVERAGE POWER IS", -K*337.5
Here is a sample run:
output on 08:49:45 06-27-1991
FRACTIONAL THICKNESS OF CUT? .5
PHI (SUGGEST A NUMBER BETWEEN 0 AND 0.2 TO STAY
IN OPTIC ZONE)? .1
AVERAGE POWER IS 37.88555
```

**Computes the curvature κ_{av} of equation 3 for the parameters of Table 1. The program is followed by a sample run.*

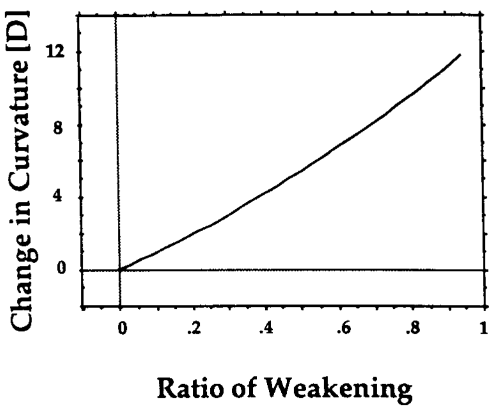

Figure 3: Graph of change in curvature versus ratio of weakening. A plot of equations 3 and 4 with the parameters of Table 1 and $E_0 = 5 \times 10^6$ dynes/cm^2. Note that the effectiveness of the cut in flattening the cornea in the optic zone increases nonlinearly with an increase in depth of cut as reflected by the concave upward shape of this graph.

cornea.) At $\phi = 0.1$, equation 3 predicts $\kappa_{av} = 43.27$ D before keratotomy and $\kappa_{av} = 37.89$ D after keratotomy, representing a 12% decrease in curvature.

We can utilize our model to obtain an estimate for E_0 by using the experimental data of Thomas et al[9] who found that an average increase in intraocular pressure of 8.2 mm Hg resulted in an average dioptric change of 0.25 to 0.50 D toward hyperopia. From equation 3 with no surgical cut ($B_1 = 0$) we obtain:

$$\kappa_{av} = -\frac{1}{a} + \frac{p(1-\nu)}{2E_0h_0} \tag{5}$$

Solving for E_0 in terms of the change in pressure Δp and the change in average curvature $\Delta\kappa_{av}$ from two measurements, we obtain:

$$E_0 = \frac{(1-\nu)}{2h_0}\frac{\Delta p}{\Delta\kappa_{av}} \tag{6}$$

Using $\Delta p = 8.2$ mm Hg $= 10933$ dynes/cm^2, and $\Delta\kappa_{av} = 0.30$ D $= 8.88 \times 10^{-4}$ mm^{-1} and the values of υ and h_0 given in Table 1, we obtain $E_0 = 5.1 \times 10^6$ dynes/cm^2.

Equations 3 and 4 show that the effectiveness of the cut in flattening the cornea in the optic zone increases nonlinearly with an increase in depth in cut (Fig 3).

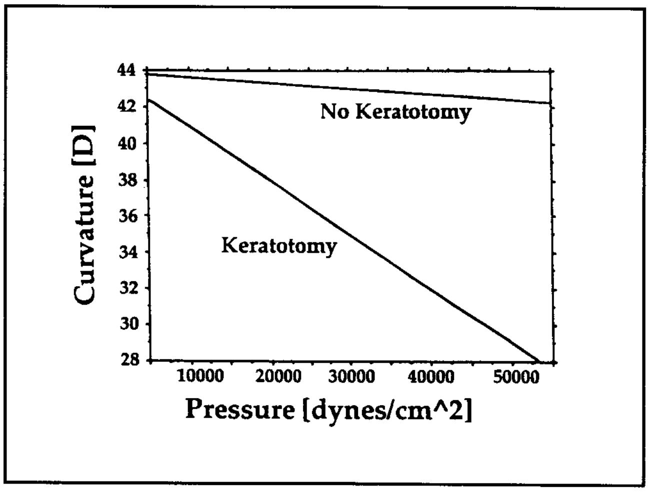

Figure 4: Graph of curvature versus intraocular pressure. A plot of equations 3 and 4 with the parameters of Table 1 and $E_0 = 5 \times 10^6$ dynes/cm^2. This graph shows that the effect of pressure changes on average curvature is greater with keratotomy than without it.

The effect of pressure changes on the average curvature κ_{av} is greater with the keratotomy than without it (Fig 4). This confirms the experimental results of Busin et al and suggests that "patients with wide fluctuations in intraocular pressure could present a higher risk for postoperative fluctuation in visual acuity."[10] The model also shows that the local curvature of a keratotomized eye is not constant across the optic zone. This suggests that refraction will generally not be uniform across the pupil.

DISCUSSION

A problem with the model is that it is unable to connect the stiffness function $B(\phi)$ with the surgical cuts of radial keratotomy in a direct manner. To model radial keratotomy, we must guess at a qualitatively reasonable function $B(\phi)$. To obtain realistic deformations, we found it necessary to assume a function $B(\phi)$ which is smaller than B_0 (the unkeratotomized value of B) outside the immediate region in which the surgical cuts are made. An extension of this work would involve choosing $B(\phi)$ on the basis of a structural model of the cornea, including dependence of $B(\phi)$ on the geometry of the surgical cuts. Such an approach, however, would lose the mathematical simplicity associated with axisymmetry.

Of course, when the surgery is axisymmetric, as is often the case in lamellar keratectomy or laser myopic keratomileusis, these caveats do not apply. These surgeries can be modeled by simply including the new (reduced) thickness of the cornea in the term $B(\phi)$. The optical result of the surgery will be given by additively combining the new corneal shape as computed from the thin-shell results and the effect of the removal of the "lens" of corneal material. In this regard, it is interesting to note that the optical effect of removing a positive lens of material from the anterior pole of the cornea (ie, making it thinner at the center of the optic zone than in the periphery) will be, in part, countered by the greater deformation of the cornea in the optic zone due to its thinning.

In the solution presented in this article, we have neglected bending effects. These effects would be important in the neighborhood of the edge of the shell, ie, at the mechanical connection between the cornea and the sclera. In fact, we have also obtained a solution including such effects, in which the scleral boundary condition was modeled as an elastic ring, thus limiting displacement and rotation at the edge of the shell. We found that at points far from the edge of the shell, ie, at points near the axis of symmetry, the shell's deformed (loaded) shape was relatively unaffected by the stiffness of the elastic ring. Since for this work we are interested in the shape of the cornea near the axis of symmetry, this shows that the presence of the support ring is not important in this model.

REFERENCES

1. Arrowsmith PN, Marks RG. Four-year update on predictability of radial keratotomy. *Refract Corneal Surg.* 1988;4:37-45.
2. Williams S, Grant W. Bending resistance in modeling the cornea as a thin shell and the effects in radial keratotomy. *ASME Biomechanics Symposium.* 1985;AMD vol 68;FED vol 21: 81-84.
3. Huang T, Bisarnsin T, Schachar RA, Black TD. Corneal curvature change due to structural alteration by radial keratotomy. *J Biomech Eng.* 1988;110:249-253.
4. Hanna KD, Jouve F, Bercovier MH, Waring GO. Computer simulation of lamellar keratectomy and laser myopic keratomileusis. *Refract Corneal Surg.* 1989;4:222-231.
5. Bryant MR, Velinsky SA. Design of keratorefractive surgical procedures: radial keratotomy. In: Ravani B, ed. *Advances in Design Automation.* New York, NY: ASME; 1989;19-1:383-391.
6. Vito RP, Shin TJ, McCarey BE. A mechanical model of the cornea: the effects of physiological and surgical factors on radial keratotomy surgery. *Refract Corneal Surg.* 1989;5:82-88.
7. Mow CC. A theoretical model of the cornea for use in studies of tonometry. *Bulletin of Math Biophysics.* 1968;30:437-453.
8. Rand RH, Lubkin SR, Howland HC. Analytical model of corneal surgery. *J Biomech Eng.* 1991;113:239-241.
9. Thomas RA, Martinez JA, Nieves J, Applegate RA. Valsalva-induced intraocular pressure effects on the corneal curvature. *Invest Ophthalmol Vis Sci.* 1991;32:1000.
10. Busin M, Yau CW, Avni I, McDonald MB, Kaufman HE. The effect of changes in intraocular pressure on corneal curvature after radial keratotomy in the rabbit eye. *Ophthalmology.* 1986;93:331-334.
11. Howland HC, Rand RH, Lubkin SR. Analytical model of corneal surgery. *Invest Ophthalmol Vis Sci.* (Abstract ARVO Annual Meeting.) 1990;31:481.
12. Bennett AG, Rabbetts RB. *Clinical Visual Objects.* 2nd ed. London: Butterworths; 1989.
13. Davson H, ed. *The Eye.* Orlando, Fla: Academic Press; 1984;1a.
14. Hoeltzel DA, Altman P, Buzard K, Choe KI. Strip extensiometry for comparison of the mechanical response of bovine, rabbit, and human corneas. *J Biomech Eng.* In press.
15. Nash IS, Green PR, Foster CS. Comparison of mechanical properties of keratoconus and normal corneas. *Exp Eye Res.* 1982;35:413-423.