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Analytical Model of Corneal Surgery

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We present a model of the human cornea in order to study the changes in its shape resulting from surgical operations (e.g., radial keratotomy). A simple closed-form solution is given for a thin linearly elastic spherical shell model of the cornea. We assume axisymmetry and isotropy in the shell surface. The surgery is modeled by permitting Young's modulus and shell thickness to depend on position. The analytical nature of the solution permits principal shell curvatures to be explicitly calculated. The model is used to investigate the effect of surgery on corneal flattening and the associated sensitivity to intraocular pressure changes.

Introduction

The cornea of the human eye is a thin shelled structure which is the principal refractive surface of the eye. It is bounded on one side by air (refractive index 1.000) and on the other by aqueous humour (refractive index 1.336). The refractive index of the cornea itself is 1.376 [10].

In recent years a number of surgical operations have been devised for altering the refractive power of the cornea, with a view towards eliminating the need for spectacles for clear vision. The most popular of these has been radial keratotomy, in which the effective elasticity of the cornea is altered by a series of 4 to 16 radial incisions centered around a clear, 3-4 mm "optical zone" at the anterior pole of the cornea. Other refractive operations on the cornea include lamellar keratectomy, where a "button" of tissue is cut from the anterior cornea, and laser myopic keratomileusis, where a lens-shaped button is ablated from the anterior cornea [5].

A major problem with refractive surgery is the lack of an accurate method for prediction of the refractive state after the surgery [1]. Most predictive methods are empirical, probably because all of the techniques have been developed along intuitive lines without reference to formal mechanical models.

Recently, finite difference methods [15] and finite element methods [2, 5, 7, 13] have been employed in an effort to rationalize the methods for predicting the mechanical, and hence refractive effects of corneal surgery.

In contrast to these numerical treatments, Mow [8] presented an analytical solution for the cornea treated as a complete sphere with point symmetry. This permitted the stresses in the corneal wall to be expressed as a function of radial position r. However, the results of this model could not be readily used to study corneal surgery, which by necessity requires that the shell properties be permitted to vary from point to point.

Mathematical Model

The cornea is modeled as a thin linearly elastic spherical shell with axisymmetry and isotropy in one shell surface. It is assumed to be loaded by a uniform hydrostatic pressure representing the effect of the aqueous humor in the anterior chamber. The surgery is modeled by permitting Young's modulus E and shell thickness h to depend upon the colatitude φ.

We have neglected bending effects. In order to evaluate their importance, we developed an approximate analytic solution which included bending effects (see [12]), the accuracy of which we checked by numerically integrating the governing differential equations. The resulting solution showed that bending effects were important only in the immediate neighborhood of the edge of the shell, i. e. at the mechanical connection between the cornea and the sclera, a region which is unimportant for this study.

The equations governing the axisymmetric spherical shell model are derived in Timoshenko and Woinowsky-Krieger [12], Chapters 14 and 16. The displacement field of a spherical shell without bending, which is loaded only by a uniform hydrostatic pressure p, may be obtained in terms of polar components v and w (positive inward) respectively representing the tangential and radial components of displacement as:

$$v=0, \quad w=-\frac{a^2p(1-\nu)}{2B(\varphi)} \tag{1}$$

where φ = colatitude (measured in a meridional plane from the pole),
a = shell radius
p = hydrostatic pressure
ν = Poisson's ratio
$B(\varphi)$ = $E(\varphi)\, h(\varphi)$ = local shell stiffness
$E(\varphi)$ = Young's modulus
$h(\varphi)$ = shell thickness

In order to model the effect of corneal surgery, we choose the colatitude dependence of the stiffness $B(\varphi) = E(\varphi)\, h(\varphi)$ to reflect the effective local structural weakening due to the surgical cuts. This includes the effects of changes in both Young's modulus E as well as in shell thickness h. It would be desirable to be able to relate the function $B(\varphi)$ directly to the geometry of the surgical cuts, but such a relation is unavailable at the present time. In lieu of such a relation, we choose to write (in the case of radial keratotomy)

$$B(\varphi)=B_0-\frac{1}{2}B_1\left[1-\cos\frac{2\pi}{\alpha}\varphi\right] \tag{2}$$

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where B_0 = stiffness before surgery, assumed independent of φ

B_1 = modulus of structural weakening

Thus after surgery, B remains unchanged ($= B_0$) at $\varphi = 0$ (the pole) and $\varphi = \alpha$ (the edge of the shell), while it decreases to $B_0 - B_1$ at the point of greatest weakening, where $\varphi = \alpha/2$.

It is expected that the main effect of the keratotomy is a flattening of the corneal surface in the optic zone, i.e., the polar corneal region which is uncut in the real eye and which, in the model, is least affected by the assumed form of $B(\varphi)$. The optic zone approximately corresponds to $0 \le \varphi \le 0.2$. In order to investigate this, we use the foregoing results to compute the curvature of the deformed shell. For the displacement field (1), the deformed shell shape may be written in x, y coordinates (where y is the axis of symmetry) as:

$$x(\varphi) = (a-w)\sin\varphi, \quad y(\varphi) = (a-w)\cos\varphi \tag{3}$$

Then the two principal curvatures at a point φ are given by the expressions [11]:

$$\kappa_1 = \frac{x'y'' - y'x''}{\left[x'^2 + y'^2\right]^{3/2}} \tag{4}$$

$$\kappa_2 = \frac{y'}{x\left[x'^2 + y'^2\right]^{1/2}} \tag{5}$$

where primes represent differentiation with respect to φ. Here κ_1 is the curvature of the generating curve, and κ_2 is the inverse of the distance from the shell to the axis of revolution, measured along a normal to the shell surface.

We (i) substitute equation (3) into equations (4) and (5), (ii) linearize the result of (*i*) in w (since the shell theory assumes small displacements), (iii) substitute equations (1), (2) into the result of (ii), and finally (iv) Taylor expand the result of (iii) in φ about $\varphi = 0$, giving:

$$\kappa_1 = -\frac{1}{a} + \frac{p(1-\nu)}{2B_0} + \frac{p(1-\nu)}{B_0}\left[\frac{\pi}{\alpha}\right]^2 \frac{B_1}{B_0}\left[1 + C\varphi^2 + \cdots\right] \tag{6}$$

$$\kappa_2 = -\frac{1}{a} + \frac{p(1-\nu)}{2B_0} + \frac{p(1-\nu)}{B_0}\left[\frac{\pi}{\alpha}\right]^2 \frac{B_1}{B_0}\left[1 + \frac{C}{3}\varphi^2 + \cdots\right] \tag{7}$$

Where

$$C = \frac{1}{2} - 2\left[\frac{\pi}{\alpha}\right]^2 + 6\left[\frac{\pi}{\alpha}\right]^2 \frac{B_1}{B_0} \tag{8}$$

In order to evaluate these curvatures, we used the following values for the physical parameters:

$$\begin{aligned} a &= 7 \text{ mm ([10], p. 57)} \\ p &= 2\times 10^4 \text{ dynes/cm}^2 \text{ ([4], p. 408)} \\ \nu &= 0.5 \\ \alpha &= \pi/3 \text{ radians ([4], p.12)} \\ E_0 &= 3\times 10^7 \text{ dynes/cm}^2 \text{ ([9], p.421)} \\ h_0 &= 0.7 \text{ mm ([4], p.12)} \\ B_0 &= E_0 h_0 \\ B_1 &= 0.9 B_0 \end{aligned} \tag{9}$$

It is convenient to express corneal curvature in diopters, which is obtained by multiplying curvature in mm $^{-1}$ by 337.5, a factor based on the difference between the refractive indices of cornea and air. For the undeformed spherical shell, both κ_1 and κ_2 equal $-1/a$, where a is the shell radius. (For convenience we will drop the minus sign on κ_1 and κ_2 in what follows.) For $a = 7$ mm, this gives $\kappa_1 = \kappa_2 = 48.21$ diopters. Numerical evaluation of equations (6)–(8) using the parameters (9) shows that the keratotomy produces a decrease in κ_1 and κ_2 in the optic zone, as desired, e.g., at $\varphi = 0.1$, equation (6) predicts $\kappa_1 = 48.13$ diopters before keratotomy, and $\kappa_1 = 46.43$ diopters after keratotomy, representing a 3.5 percent decrease in curvature κ_1. Equation (7) gives the corresponding values for κ_2 as 48.13 diopters before keratotomy and 46.70 diopters after keratotomy, representing a 3.0 percent decrease in κ_2. The values of κ_1 and κ_2 are generally found to be close to each other for points in the optic zone.

Discussion and Conclusions

The model of the cornea presented here gives realistic quantitative changes in corneal curvature in the optic zone associated with keratotomy surgery, for realistic values of the parameters.

However, the model is unable to connect the stiffness function $B(\varphi)$ with the surgical cuts of radial keratotomy in a direct manner. In order to model radial keratotomy, we must guess at a qualitatively reasonable function $B(\varphi)$. In order to obtain realistic deformations, we found it necessary to assume a function $B(\varphi)$ which is smaller than B_0 (the unkeratotomized value of B) outside the immediate region in which the surgical cuts are made. An extension of this work would involve choosing $B(\varphi)$ on the basis of a structural model of the cornea, including dependence of $B(\varphi)$ on the geometry of the surgical cuts. Such an approach, however, would lose the mathematical simplicity associated with axisymmetry. By modeling the surgery as axisymmetric, we have essentially averaged the surgical effects around the symmetry axis of the shell. This approximation is expected to work best away from the immediate neighborhood of the surgery, for example in the optic zone.

Of course, when the surgery is axisymmetric, as is often the case in lamellar keratectomy or laser myopic keratomileusis, these caveats do not apply. These surgeries can be modelled by simply including the new (reduced) thickness of the cornea in the term $B(\varphi)$. The optical results of the surgery will be given by additively combining the new corneal shape as computed from the thin shell results and the effect of the removal of the "lens" of corneal material. In this regard, it is interesting to note that the optical effect of removing a positive lens of material from the pole of the cornea (i.e., making it thinner at the optic zone than the periphery) will be, in part, countered by the greater deformation of the cornea in the optic zone due to its thinning.

Using the analytical results of the model, we are able to draw certain conclusions about what the model has to say about corneal surgery. The model shows that the keratotomy is more effective, as measured by curvature changes in the optic zone, when the ratio of weakening, B_1/B_0, is closer to unity. This corresponds to the observation that the surgical cuts are most effective if they are almost all the way through the cornea. For example, Werner [14] states that the radial keratotomy technique has evolved to incorporate incisions which go through 90 to 95 percent of the corneal thickness. In fact, equations (6)-(8) show that the effectiveness of the cut in flattening the cornea in the optic zone increases nonlinearly with an increase in depth of cut, i.e., the graph of change in curvature resulting from keratotomy versus ratio of weakening B_1/B_0 is concave upward.

Moreover, the model shows that the effect of pressure changes on curvatures κ_1 and κ_2 is greater with the keratotomy than without it. For example, when p was increased from 20000 dynes/cm^2 to 40000 dynes/cm^2, the unkeratotomized cornea at $\varphi = 0.1$ changed curvature κ_1 by 0.16 percent, while in the keratotomized eye κ_1 changed by 3.8 percent. This confirms the experimental results of Busin et al. [3] and suggests that "patients with wide fluctations in intraocular pressure could present a higher risk for postoperative fluctuation in visual acuity" [3].

It is interesting to note that while the model presented in this work is less realistic than the finite element model presented by Vito et al. [13], certain features of the results compare favorably. In particular, our observation that the graph of change in curvature versus increase in depth of cut (represented by a local decrease in B) is concave upward, is consistent with Fig. 6 in [13]. Also, Vito et al. [13] showed that increasing the intraocular pressure from 15 mm Hg (= 20000 dynes/cm^2) to 25 mm Hg (= 33000 dynes/cm^2) caused a change in curvature of about 4 percent, a result which is approximately in agreement with our results given in the preceding paragraph.

The model also shows that the local curvature of a keratotomized eye is not constant across the optic zone. This suggests that refraction will generally not be uniform across the pupil.

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