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# Nonlinear Effects on Coexistence Phenomenon in Parametric Excitation

LESLIE NG and RICHARD RAND
*Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, U.S.A.*

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**Abstract.** We investigate the effect of nonlinearites on a parametrically excited ordinary differential equation whose linearization exhibits the phenomenon of coexistence. The differential equation studied governs the stability of a mode of vibration in an unforced conservative two degree of freedom system used to model the free vibrations of a thin elastica. Using perturbation methods, we show that at parameter values corresponding to coexistence, nonlinear terms can cause the origin to become nonlinearly unstable, even though linear stability analysis predicts the origin to be stable. We also investigate the bifurcations associated with this instability.

**Keywords:** Parametric excitation, coexistence, averaging, bifurcation.

## 1. Introduction

In this work we look at the following parametrically excited Ordinary Differential Equation (ODE):

$$\left(1 - \frac{\varepsilon}{2}\cos 2t\right)\ddot{x} + \varepsilon \sin 2t\, \dot{x} + cx + \varepsilon\alpha x^2 = 0. \tag{1}$$

When $\alpha = 0$, Equation (1) arises in the study of the dynamics of a thin elastica which was the subject of the Ph.D. Thesis of Cusumano [1]. Also when $\alpha = 0$, Equation (1) is a form of Ince's equation and exhibits the phenomenon of coexistence [3]. By taking $\alpha > 0$, we add a nonlinear spring to the physical model previously studied in [1, 4], permitting us to investigate the effect of nonlinearities on a system exhibiting coexistence.

We start by reviewing the phenomenon of coexistence in Ince's equation. Next, we derive Equation (1) from a model proposed by Cusumano [1] and show how the quadratic nonlinearity may enter in the equation. Finally, we investigate the effect of the quadratic nonlinearity in Equation (1) by using pertubation methods.

## 2. Review of Coexistence and Ince's Equation

A well-known ordinary differential equation with periodic coefficients is Mathieu's equation:

$$\ddot{z} + (\delta + \varepsilon \cos 2t)z = 0. \tag{2}$$

For given values of the parameters $\delta$ and $\varepsilon$, either all the solutions of are bounded (*stable*) or an unbounded solution exists (*unstable*). The curves separating the stable and unstable regions in the $\delta$–$\varepsilon$ plane are known as *transition curves* (Figure 1). The instability intervals which emanate out of the $\delta$ axis at values $\delta = n^2$ for $n = 1, 2, 3, \ldots$ are commonly referred to as *resonance tongues*.

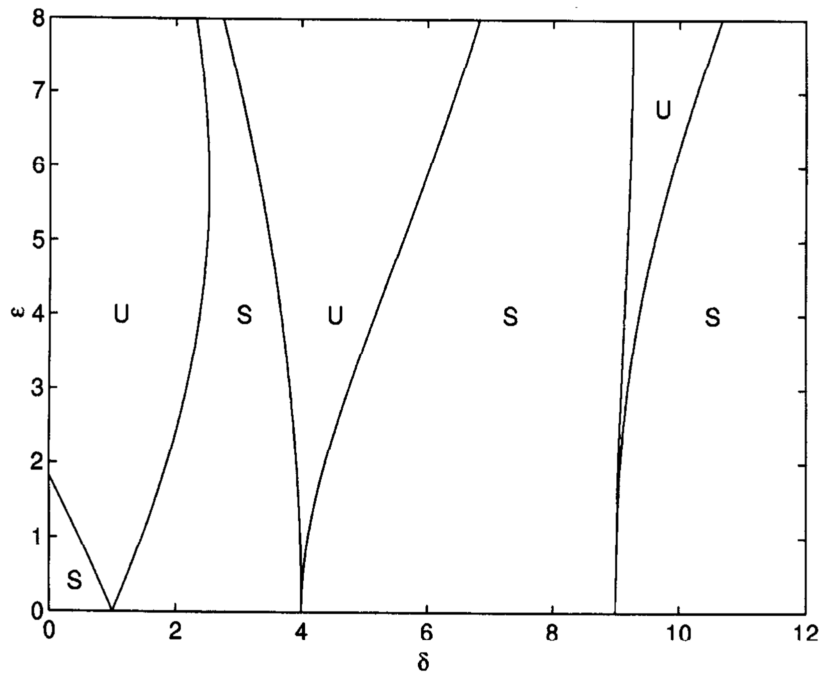


*Figure 1.* Transition curves separating regions of stability from instability in Mathieu's equation.

The presence of resonance tongues is a generic feature of differential equations with periodic coefficients. One phenomenon that does *not* occur in Mathieu's equation but which may occur in other differential equations with periodic coefficients is *coexistence*. The phenomenon of coexistence involves the disappearance of resonance tongues that would normally be present. In systems that exhibit coexistence, the two transitions curves that would normally define a resonance tongue coincide and the tongue closes up. An example of an equation that may exhibit coexistence is Ince's equation [3]:

$$(1 + a\cos 2t)\,\ddot{x} + b\sin 2t\;\dot{x} + (c + d\cos 2t)\,x = 0. \tag{3}$$

Note that Mathieu's equation is a special case of Ince's equation which turns out not to exhibit coexistence. We know from Floquet theory that since the periodic coefficients in Equation (3) have period $\pi$, the solutions on the transition curves will be periodic with period $\pi$ or $2\pi$. Let us assume that $a$, $b$ and $d$ depend on $\varepsilon$. If $a = b = d = 0$ when $\varepsilon = 0$, then Equation (3) becomes:

$$\ddot{x} + cx = 0, \tag{4}$$

which has solutions of period $2\pi/\sqrt{c}$. These correspond to solutions of period $\pi$ or $2\pi$ when $2\pi/\sqrt{c} = \frac{2\pi}{n}$ so we would expect resonance tongues in the $c$–$\varepsilon$ plane to emanate from the points $c = n^2$, $n = 1, 2, 3, \ldots$ on the $c$-axis.

We use the method of harmonic balance to investigate what happens to the tongues of instability in Ince's equation [3, 7]. Since the transition curves are characterized by having solutions of period $\pi$ or $2\pi$, we expand the solution $x$ in a Fourier series:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt. \tag{5}$$

Substituting Equation (5) into Equation (3), simplifying and collecting trigonometric terms, we obtain four sets of algebraic equations on the coefficients $a_n$ and $b_n$. Each set deals exclusively with $a_{\text{even}}$, $b_{\text{even}}$, $a_{\text{odd}}$ and $b_{\text{odd}}$. Also, each set is homogeneous and of infinite order so for a nontrivial solution, the determinants must vanish. The four infinite determinants are:

$$a_{\text{even}}: \begin{vmatrix} c & \frac{d}{2}-b-2a & 0 & 0 & 0 & \\ d & c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ 0 & \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \cdots \\ 0 & 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \\ 0 & 0 & 0 & \frac{d}{2}+3b-18a & c-64 & \\ & & & \cdots & & \end{vmatrix}, \tag{6}$$

$$b_{\text{even}}: \begin{vmatrix} c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \cdots \\ 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \\ 0 & 0 & \frac{d}{2}+3b-18a & c-64 & \\ & & \cdots & & \end{vmatrix}, \tag{7}$$

$$a_{\text{odd}}: \begin{vmatrix} c-1+\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \cdots \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{vmatrix}, \tag{8}$$

$$b_{\text{odd}}: \begin{vmatrix} c-1-\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \cdots \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{vmatrix}. \tag{9}$$

The notation in these determinants can be simplified by setting (after Magnus and Winkler, 'Hill's Equation' [3]):

$$Q(m) = \frac{d}{2} + bm - 2am^2, \tag{10}$$

$$P(m) = Q\left(m - \frac{1}{2}\right) = \frac{d + b(2m-1) - a(2m-1)^2}{2}. \tag{11}$$

Using this notation for $Q(m)$ and $P(m)$, taking the four infinite determinants in Equations (6–9) and setting them to zero gives:

$$a_{\text{even}}: \begin{vmatrix} c & Q(-1) & 0 & 0 & 0 & \\ 2Q(0) & c-4 & Q(-2) & 0 & 0 & \\ 0 & Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & 0 & Q(2) & c-36 & Q(-4) & \\ 0 & 0 & 0 & Q(3) & c-64 & \\ & & & \cdots & & \end{vmatrix} = 0, \tag{12}$$

$$b_{\text{even}}: \begin{vmatrix} c-4 & Q(-2) & 0 & 0 & \\ Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & Q(2) & c-36 & Q(-4) & \\ 0 & 0 & Q(3) & c-64 & \\ & & \cdots & & \end{vmatrix} = 0, \tag{13}$$

$$a_{\text{odd}}: \begin{vmatrix} c-1+P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(3) & c-25 & P(-3) & \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{vmatrix} = 0, \tag{14}$$

$$b_{\text{odd}}: \begin{vmatrix} c-1-P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(2) & c-25 & P(-3) & \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{vmatrix} = 0. \tag{15}$$

Comparison of the determinants in Equation (12) and Equation (13) shows that if the first row and first column of Equation (12) are removed, then the remainder of Equation (12) is identical to Equation (13). The significance of this observation is that if any one of the off-diagonal terms vanishes, that is if $Q(m) = 0$ for some integer $m$ (positive, negative or zero), then coexistence can occur and an infinite number of possible tongues of instability will not occur.

In order to understand how this works, suppose $Q(2) = 0$. Then we may represent Equations (12) and (13) symbolically as follows:

$$a_{\text{even}}: \begin{vmatrix} X & X & 0 & 0 & 0 & \\ X & X & X & 0 & 0 & \\ 0 & X & X & X & 0 & \cdots \\ 0 & 0 & Q(2) & X & X & \\ 0 & 0 & 0 & X & X & \\ & & & \cdots & & \end{vmatrix} = 0, \tag{16}$$

$$b_{\text{even}}: \begin{vmatrix} X & X & 0 & 0 & \\ X & X & X & 0 & \cdots \\ 0 & Q(2) & X & X & \\ 0 & 0 & X & X & \\ & & \cdots & & \end{vmatrix} = 0, \tag{17}$$

where we have used the symbol $X$ to represent a term which is nonzero. The vanishing of $Q(2)$ 'disconnects' the lower (infinite) portion of these equations from the upper (finite) portion. There are now two possible ways in which to satisfy these equations with $Q(2) = 0$:

1. For a nontrivial solution to the lower (infinite) portion, the (disconnected, infinite) determinant must vanish. Since this determinant is identical for both the $a$'s and $b$'s, coexistence is present and the associated tongues emanating from $c = 36, 64, \ldots$ do not occur. The coefficients $a_6, a_8, a_{10}, \ldots$ and $b_6, b_8, b_{10}, \ldots$ will not in general vanish. In this case the

upper portion of the determinant will not vanish in general, and the coefficients $a_0$, $a_2$, $a_4$, and $b_2$, $b_4$ will not be zero because they depend respectively on $a_6$ and $b_6$.

2. Another possibility is that the infinite determinant of the lower portion is not zero, requiring that the associated $a_{\text{even}}$ and $b_{\text{even}}$ coefficients vanish. With these $a$'s and $b$'s zero, the upper portion of the system becomes independent of the lower, and for a nontrivial solution for $a_0$, $a_2$, $a_4$, $b_2$ and $b_4$, the upper portion of both determinants must vanish. For Equation (16) this involves a $3 \times 3$ determinant and yields a cubic on $c$, while for Equation (17) this involves a $2 \times 2$ determinant and gives a quadratic on $c$. Together these yield five expressions for $c$ in terms of the other parameters of the problem, which, if real, correspond to the five transition curves. One of these passes through the $c$-axis at $c = 0$, and the other four produce tongues of instability emanating from $c = 4$ and $c = 16$ respectively.

A similar situation occurs for Equations (14) and (15). If $P(m) = 0$ for some integer $m$ (positive, negative or zero) then only a finite number of tongues will occur from amongst the infinite set of tongues which emanate from the points $c = (2n - 1)^2$, $n = 1, 2, 3, \ldots$ on the $c$-axis.

As an example we take Equation (1) when $\alpha = 0$. It is in the form of Ince's equation with the parameter values:

$$a = -\frac{\varepsilon}{2}, \quad b = \varepsilon, \quad d = 0. \tag{18}$$

For the parameter values in Equations (18), the polynomials $Q(m)$ and $P(m)$ from Equations (19) and (20) become

$$Q(m) = \varepsilon m + \varepsilon m^2, \tag{19}$$

$$P(m) = \frac{2\varepsilon(2m - 1) + \varepsilon(2m - 1)^2}{4}. \tag{20}$$

For Equations (19) and (20), it is easy to show that $Q(m) = 0$ when $m = 0, -1$ and $P(m) = 0$ when $m = \pm 1/2$. Substituting $Q(0) = 0$ and $Q(-1) = 0$ into Equation (12), we see that the element $c$ in the upper left corner of Equation (12) becomes disconnected from the rest of the infinite determinant, which is itself identical to the infinite determinant of Equation (13). From this we can conclude that $c = 0$ is a transition curves and all the even tongues disappear. Because $P(m)$ does not have integer roots, we can also conclude that the system has an infinite number of odd tongues.

Some other examples of systems exhibiting coexistence are given in [2, 5, 9, 7, 10].

## 3. Derivation of Differential Equation

In this section we derive Equation (1) from a model proposed by Cusumano [1]. In his thesis, Cusumano [1] studied the dynamics of a thin elastica. He showed that complicated dynamics result and that a mode of vibration exists which involves both bending and torsional modes. Figure 2 shows some of the modes of vibration of a thin elastica. To get a better understanding of the dynamics, Cusumano [1] examined the simplified two degree of freedom model shown in Figure 3.

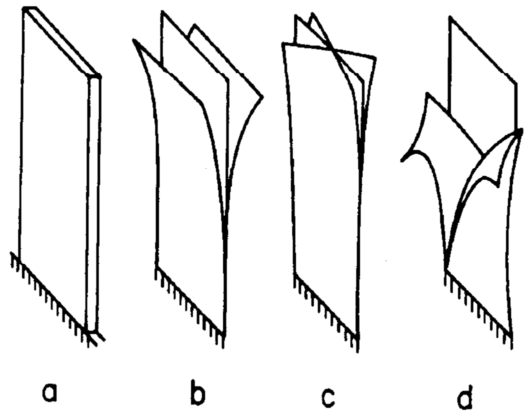


*Figure 2.* Thin elastica. (a) Undeformed; (b) bending mode; (c) torsional mode; (d) nonlocal mode, involving both bending and torsion. (Pak et al. [4]).

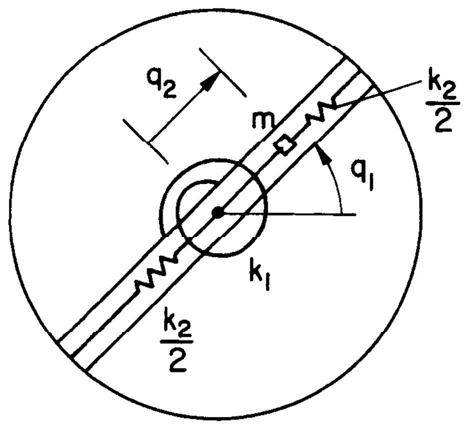


*Figure 3.* Simplified two degree of freedom model of an elastica (Cusumano [1]).

In the simplified model, the rotational motion due to coordinate $q_1$ is associated with the torsional motion of the elastica and the rectilinear deflection due to $q_2$ is associated with the bending motion. Pak et al. [4] investigated the different modes of vibration for the system in Figure 3. They found that the stability of the bending mode is governed by an equation of the form of Equation (1) with $\alpha = 0$.

We consider a system similar to the one shown in Figure 3 except with a nonlinear torsional spring. Instead of using a linear force-displacement relation as in Figure 3:

$$f = k_1 q_1. \tag{21}$$

We use the following nonlinear force-displacement relation with an additional quadratic term:

$$f = k_{11} q_1 + k_{12} q_1^2. \tag{22}$$

If we set $k_{12} = 0$ and $k_{11} = k_1$, we get back the system in Figure 3. Note that the result of adding the quadratic term is to add some asymmetry in the torsional spring. Much of the derivation presented here follows the paper by Pak et al. [4]. We begin by writing the kinetic energy, $T$, and the potential energy, $V$, for the system:

$$T = \frac{m}{2}[q_2^2(q_1')^2 + (q_2')^2] + \frac{J}{2}(q_1')^2, \tag{23}$$

$$V = \frac{1}{2}\left[k_{11}q_1^2 + \frac{2}{3}k_{12}q_1^3 + k_2q_2^2\right]. \tag{24}$$

Using the rescalings

$$x = \sqrt{\frac{Jk_2}{m}}q_1, \quad y = \sqrt{k_2}q_2, \quad t = \sqrt{\frac{k_2}{m}}\tau, \tag{25}$$

the following Lagrangian $L$ can be obtained:

$$L = \frac{1}{2}(1+\gamma y^2)\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\left(\kappa_1 x^2 + \frac{2}{3}\kappa_2 x^3 + y^2\right), \tag{26}$$

where

$$\gamma = \frac{m}{Jk_2}, \quad \kappa_1 = \frac{k_{11}/J}{k_2/m}, \quad \kappa_2 = k_{12}\left(\frac{m}{Jk_2}\right)^{3/2}. \tag{27}$$

Applying Lagrange's equations to Equation (26) gives the equations of motion

$$(1+\gamma y^2)\ddot{x} + 2\gamma y\dot{y}\dot{x} + \kappa_1 x + \kappa_2 x^2 = 0, \tag{28}$$

$$\ddot{y} - \gamma\dot{x}^2 y + y = 0. \tag{29}$$

Note that the $x$-mode, $y \equiv 0$, and the $y$-mode, $x \equiv 0$, are exact solutions to Equations (28) and (29). To investigate the stability of the $y$-mode, $x \equiv 0$, we linearize Equations (28) and (29) about the exact solution by setting

$$x = 0 + \mu\hat{x}, \quad y = A\sin t + \mu^{3/2}\hat{y}, \tag{30}$$

where $\mu$ is a small parameter. The factor $\mu^{3/2}$ in Equations (30) is chosen so that the scalings come out appropriately. Substituting Equations (30) into Equations (28) and (29) and Taylor expanding in $\mu$ gives

$$\mu(1+\gamma A^2\sin^2 t)\ddot{\hat{x}} + \mu\gamma A^2\sin 2t\ \dot{\hat{x}} + \mu\kappa_1\hat{x} + \mu^2\kappa_2\hat{x}^2 + O(\mu^{5/2}) = 0, \tag{31}$$

$$\mu^{3/2}\ddot{\hat{y}} + \mu^{3/2}\hat{y} + O(\mu^2) = 0. \tag{32}$$

Note that Equation (31) is uncoupled from $\hat{y}$ up to $O(\mu^2)$. Taking terms up to $O(\mu^2)$ in Equation (31), setting $\mu = 1$ and dropping hats gives

$$(1+\gamma A^2\sin^2 t)\ddot{x} + \gamma A^2\sin 2t\ \dot{x} + \kappa_1 x + \kappa_2 x^2 = 0. \tag{33}$$

Expanding the trigonometric term in Equation (33) gives

$$\left(1 + \frac{\gamma A^2}{2} - \frac{\gamma A^2}{2}\cos 2t\right)\ddot{x} + \gamma A^2\sin 2t\dot{x} + \kappa_1 x + \kappa_2 x^2 = 0. \tag{34}$$

Finally, dividing Equation (34) by $1 + (\gamma A^2/2)$ and taking

$$\varepsilon = \frac{\gamma A^2}{1+\frac{\gamma A^2}{2}}, \quad c = \frac{\kappa_1}{1+\frac{\gamma A^2}{2}}, \quad \varepsilon\alpha = \frac{\kappa_2}{1+\frac{\gamma A^2}{2}}, \tag{35}$$

we have obtained Equation (1):

$$\left(1 - \frac{\varepsilon}{2}\cos 2t\right)\ddot{x} + \varepsilon \sin 2t\, \dot{x} + cx + \varepsilon\alpha x^2 = 0.$$

## 4. Derivation of Coexistence Curve

We obtain a series expansion for small $\varepsilon$ of the curve of coexisting solutions emanating out of $c = 4$ in Equation (1) when $\alpha = 0$. The system exhibits coexistence for the even resonance tongues so the curves of coexisting solutions can be computed from the infinite determinant for $b_{\text{even}}$ (Equation (7)). We start by taking a finite truncation of the infinite determinant for $b_{\text{even}}$. If we take a $4 \times 4$ truncation we get

$$b_{\text{even}}: \begin{vmatrix} c-4 & \frac{d}{2}-2b-8a & 0 & 0 \\ \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 \\ 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a \\ 0 & 0 & \frac{d}{2}+3b-18a & c-64 \end{vmatrix} = 0. \tag{36}$$

It is a straight-forward computation to obtain a single equation for the determinant of Equation (36). To get an approximation for the curve of coexisting solutions out of $c = 4$, we Taylor expand $c$ in a powers series in $\varepsilon$ about $c = 4$:

$$c = 4 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + \cdots. \tag{37}$$

Recall for Equation (1), the parameter values are

$$a = -\frac{\varepsilon}{2}, \quad b = \varepsilon, \quad d = 0.$$

Substituting these values for $a$, $b$, $c$ and $d$ into the single equation obtained from Equation (36) and Taylor expanding in $\varepsilon$ we get

$$\begin{aligned} &-23040c_1\varepsilon + (-23040c_2 + 3024c_1^2 - 7680)\varepsilon^2 \\ &\quad + (-23040c_3 + 6048c_1c_2 - 104c_1^3 + 4256c_1)\varepsilon^3 + \cdots = 0. \end{aligned} \tag{38}$$

Requiring terms of $O(\varepsilon)$ in Equation (38) to be zero gives $c_1 = 0$. Using this value for $c_1$ and requiring terms of $O(\varepsilon^2)$ to be zero gives $c_2 = -1/3$. Using these values for $c_1$ and $c_2$, higher order terms can also be obtained in a similar fashion. Of course, obtaining higher order terms also requires taking more terms in the truncation of the determinant for $b_{\text{even}}$. To $O(\varepsilon^2)$, the series expansion for the curve of coexisting solutions emanating out of $c = 4$ is

$$c = 4 - \frac{1}{3}\varepsilon^2 + \cdots. \tag{39}$$

## 5. Effect of a Quadratic Nonlinearity

We use the method of averaging to investigate the effect of the quadratic nonlinearity in Equation (1). For more on the method of averaging, see [6, 8]. First, note that $\alpha$ can be rescaled out of Equation (1). We assume $\alpha > 0$, so without loss of generality we set $\alpha = 1$. From our results for $\alpha = 0$, we know that the Equation (1) exhibits coexistence and the tongues of instability that emanate out of $c = n^2$ for $n$ even vanish. To perturb off a resonance where coexistence occurs, we set

$$c = 4 + \varepsilon^2 c_2. \tag{40}$$

Substituting Equation (40) and $\alpha = 1$ into Equation (1) we get:

$$\left(1 - \frac{\varepsilon}{2}\cos 2t\right)\ddot{x} + \varepsilon \sin 2t\, \dot{x} + (4 + \varepsilon^2 c_2)x + \varepsilon x^2 = 0 \tag{41}$$

For small values of $\varepsilon$, we can apply the method of averaging to Equation (41). See Appendix A for details of the averaging calculation. To first order, the averaging procedure does not produce any terms in the slow-flow equations. Going to second order in the averaging calculation results in the following slow-flow equations:

$$\dot{R} = \frac{\varepsilon^2 R^2 \sin\psi}{48}, \tag{42}$$

$$\dot{\psi} = \frac{\varepsilon^2\left(24c_2 - 5R^2 + 6R\cos\psi + 8\right)}{96}. \tag{43}$$

We start by looking for equilibria in the slow-flow. Note that $R = 0$ is always an equilibrium point. Setting the RHS of Equation (42) to zero requires $\sin\psi = 0$ so $\psi = 0$ or $\psi = \pi$. Substituting $\psi = 0$ and $\psi = \pi$ into the RHS of Equation (43) and setting it to zero gives

$$24c_2 - 5R^2 + 6R + 8 = 0, \tag{44}$$

$$24c_2 - 5R^2 - 6R + 8 = 0. \tag{45}$$

Equations (44) and (45) differ only by the sign of $R$ term and both have the same discriminant. For real roots, we require the discriminant to be positive which results in

$$36 + 20(24c_2 + 8) \geq 0 \tag{46}$$

This gives the condition $c_2 \geq -49/120$. When $c_2 = -49/120$, a pair of equilbria are created at $R = 3/5$, $\psi = 0$. Also note that for $R = 0$ both Equations (44) and (45) are satisfied when $c_2 = -1/3$. This value of $c_2$ corresponds with the perturbation expansion for the curve of coexisting solutions emanating from $c = 4$ when $\alpha = 0$ (Equation (39)). Figure 4 shows a bifurcation diagram of the slow-flow equilibria by plotting Equations (44) and (45).

For $c_2 < -49/120$, the origin is the only equilbrium point. A pair of equilibria are created when $c_2 = -49/120$ and for $-49/120 < c_2 < -1/3$ there are two nonorigin equilibria with $\psi = 0$. At $c_2 = -1/3$, one of these equilibria goes through the origin (recall that the origin is always an equilibrium point) and $\psi$ for that equilibria changes from $\psi = 0$ to $\psi - \pi$. For $c_2 > -1/3$, there are two nonorigin equilibria, one with $\psi = 0$ and one with $\psi = \pi$.

To investigate the nature of the bifurcations, it is more convenient to look at the slow-flow equations in Cartesian coordinates. Transforming to Cartesian coordinates $u = R\cos\psi$, $v = -R\sin\psi$, Equations (42) and (43) become

$$\dot{u} = \frac{\varepsilon^2 v\left[24c_2 + 8 + 4u - 5(u^2 + v^2)\right]}{96}, \tag{47}$$

$$\dot{v} = \frac{\varepsilon^2\left[5u(u^2 + v^2) - 6(u^2 + v^2) + 4v^2 - 24c_2 u - 8u\right]}{96}. \tag{48}$$

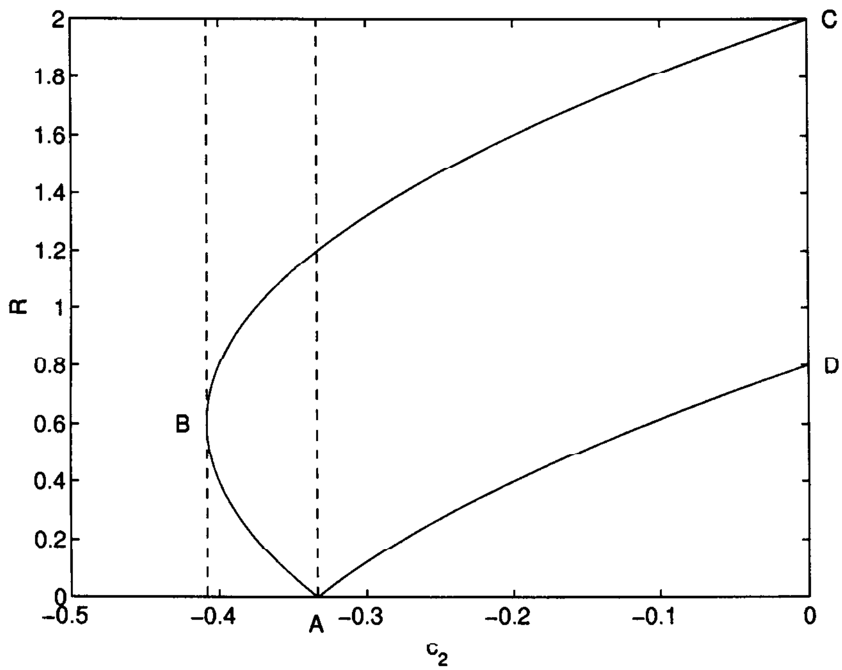


*Figure 4.* Bifurcation diagram of slow-flow equilibria. Dashed lines correspond to $c_2 = -1/3$ and $c_2 = -49/120$. Here $c_2$ is related to $c$ by $c = 4 + c_2\varepsilon^2$. $c_2 = -1/3$ corresponds to coexistence. The branch AB corresponds to saddles with $\psi = 0$. The branch BC corresponds to centers with $\psi = 0$. The branch AD corresponds to saddles with $\psi = \pi$. The origin, $R = 0$, is a center except at point A.

Equations (47) and (48) have the first integral:

$$\frac{5v^4 + (10u^2 - 8u - 48c_2 - 16)v^2 + 5u^4 - 8u^3 - (48c_2 + 16)u^2}{4} = K. \tag{49}$$

Using Equation (49), we can investigate the stability of the equilibria by plotting invariant curves of the system. Figures 5–9 show the sequence of invariant curves for the system as $c_2$ is increased. Figures 5, 7 and 9 are representative of the system in the regions $c_2 < -49/120$, $-49/120 < c_2 < -1/3$ and $c_2 > -1/3$ respectively. Figures 6 and 8 correspond to parameter values where bifurcations occur ($c_2 = -49/120$ and $c_2 = -1/3$).

For $c_2 < -49/120$, the only equilibria is the origin which is a center. At $c_2 = -49/120$, a saddle-center bifurcation occurs where a saddle and a center are created. As $c_2$ approaches the value $c_2 = -1/3$, the region of stability around the origin gets smaller as the saddle created in the saddle-center bifurcation moves toward the center at the origin. At the critical value $c_2 = -1/3$, the saddle coalesces with the origin, resulting in a degenerate, unstable equilibrium. Figure 10 shows a blow-up of the invariant curves around the origin when $c_2 = -1/3$.

As $c_2$ increases and goes through $c_2 = -1/3$, the saddle moves through the origin and the saddle and center created in the saddle-center bifurcation are now on opposite sides of the origin which remains a center. As $c_2$ further increases, these equilibria move farther away from the origin.

As $c_2$ approaches the value $c_2 = -1/3$ from either side, the region of stability around the origin gets smaller as a saddle and center come together. The value $c_2 = -1/3$ corresponds to coexistence. Linear stability analysis predicts that the origin is stable along the curve of

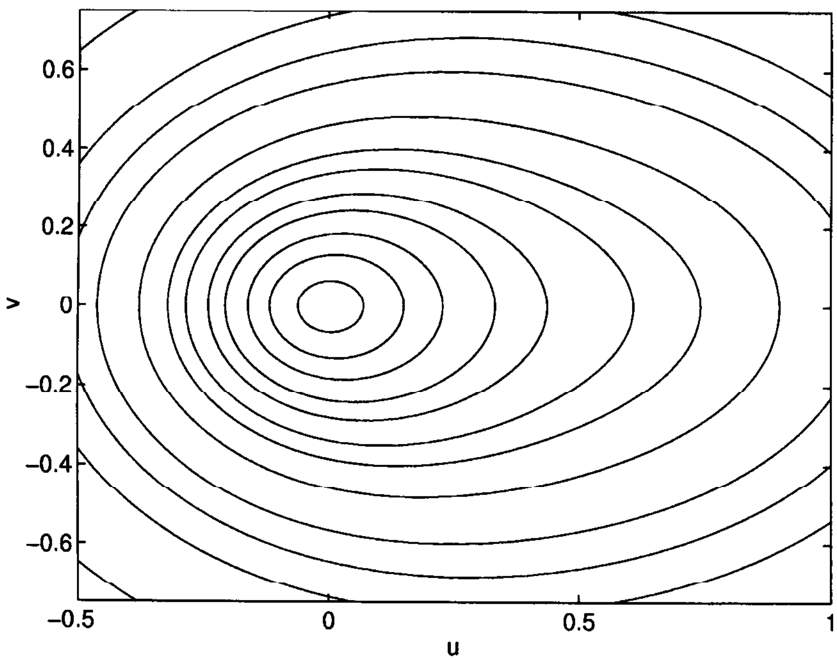


*Figure 5.* Invariant curves for $c_2 = -0.43$.

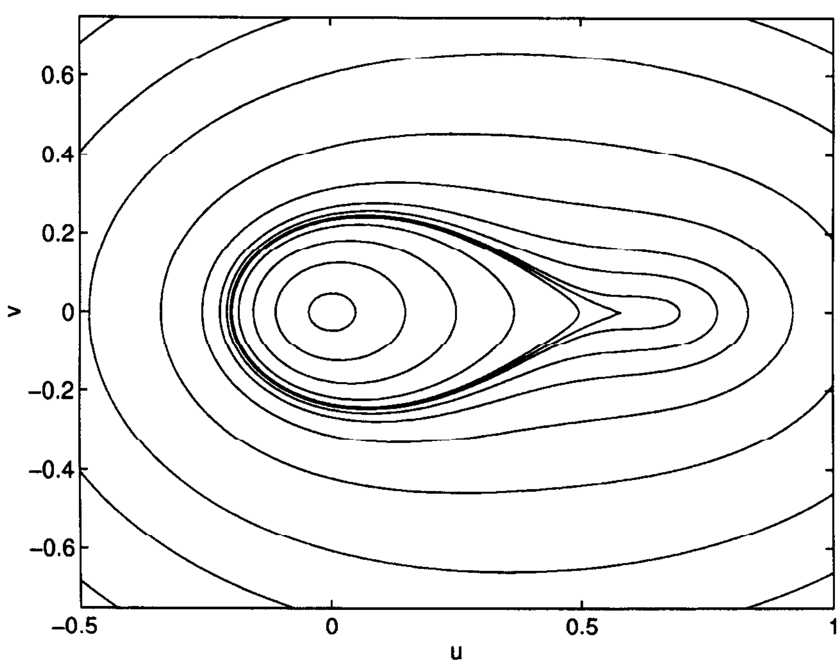


*Figure 6.* Invariant curves for $c_2 = -49/120$. Note the saddle-center bifurcation at $u = 3/5$, $v = 0$.

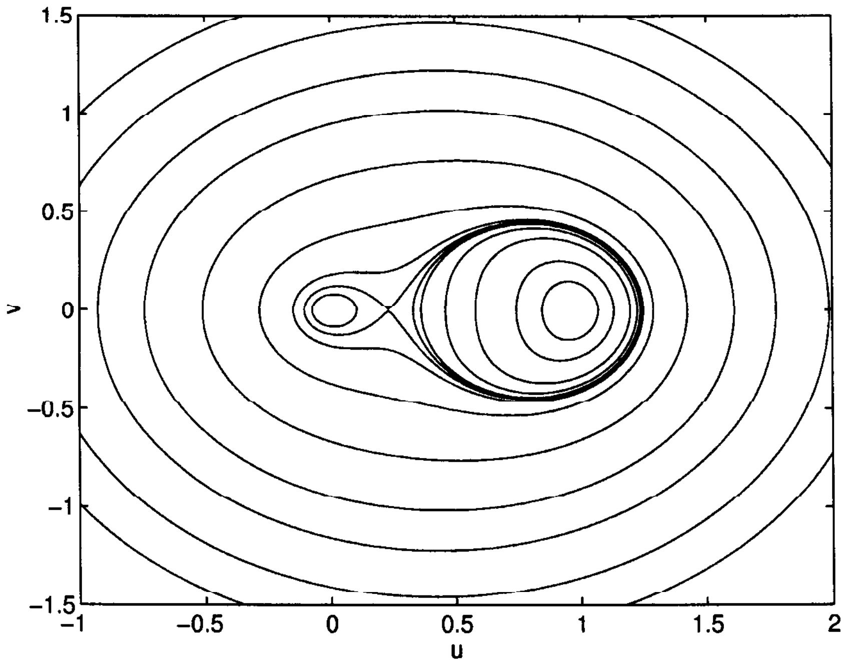


*Figure 7.* Invariant curves for $c_2 = -0.38$.

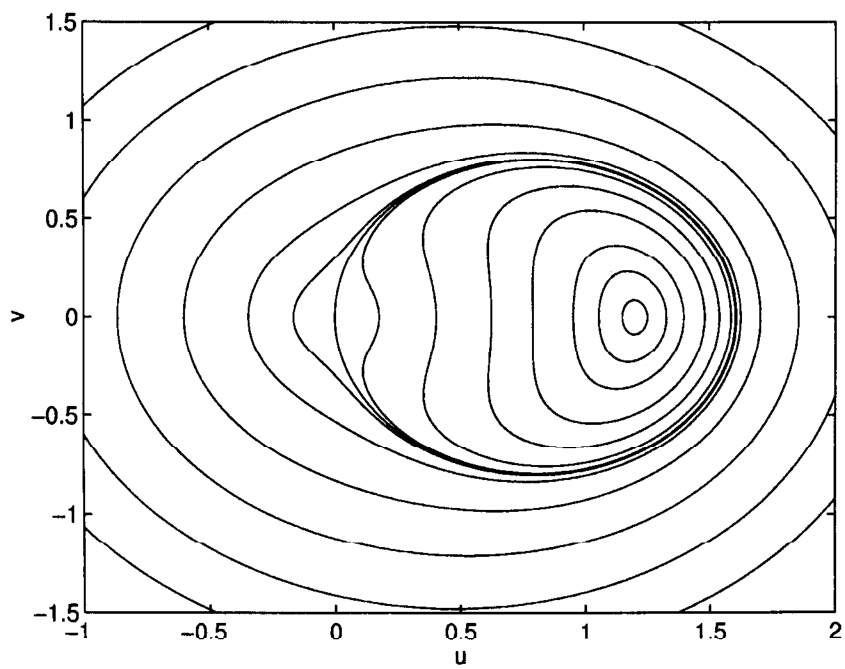


*Figure 8.* Invariant curves for $c_2 = -1/3$. Note the origin is a degenerate equilibrium point.

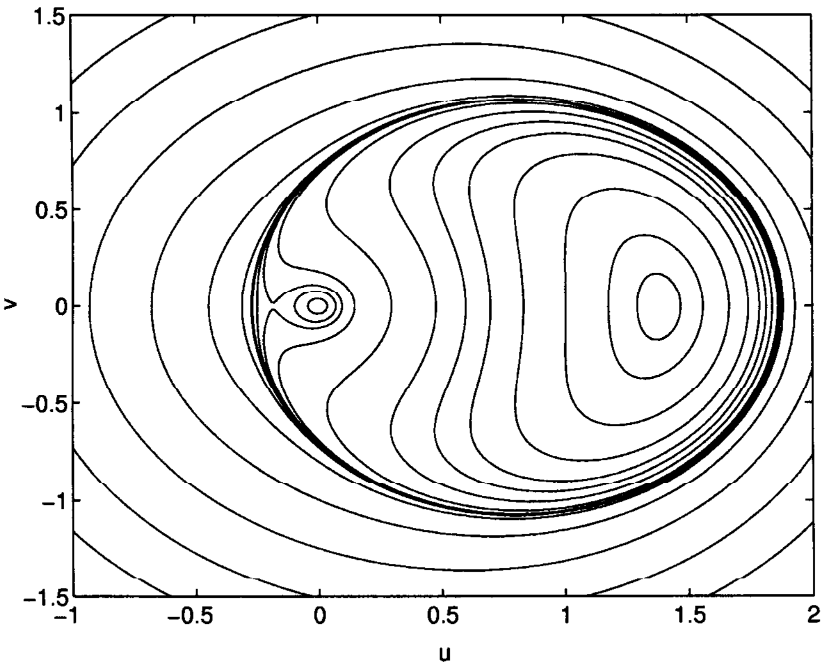


*Figure 9.* Invariant curves for $c_2 = -0.28$.

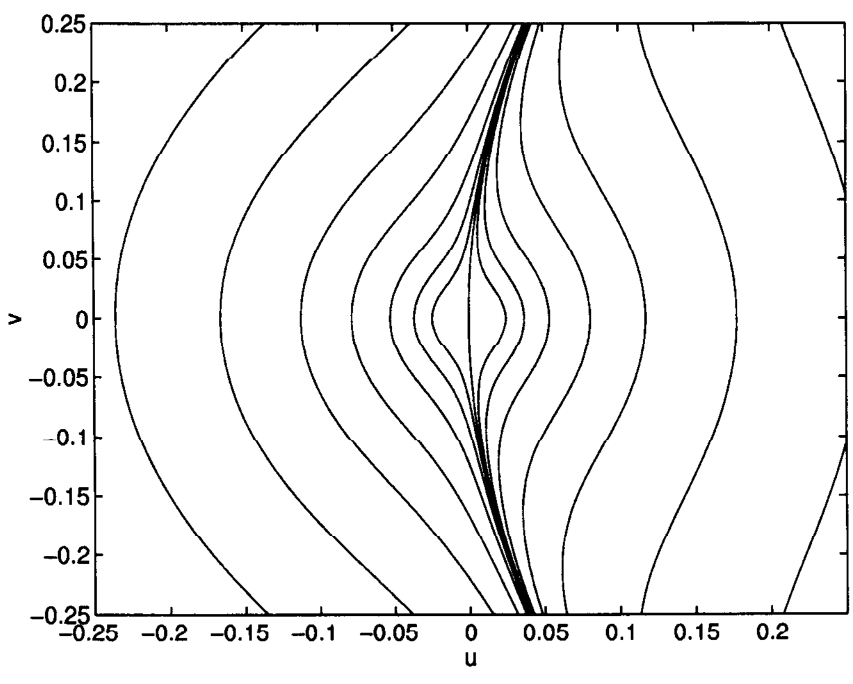


*Figure 10.* Invariant curves for $c_2 = -1/3$ near the origin.

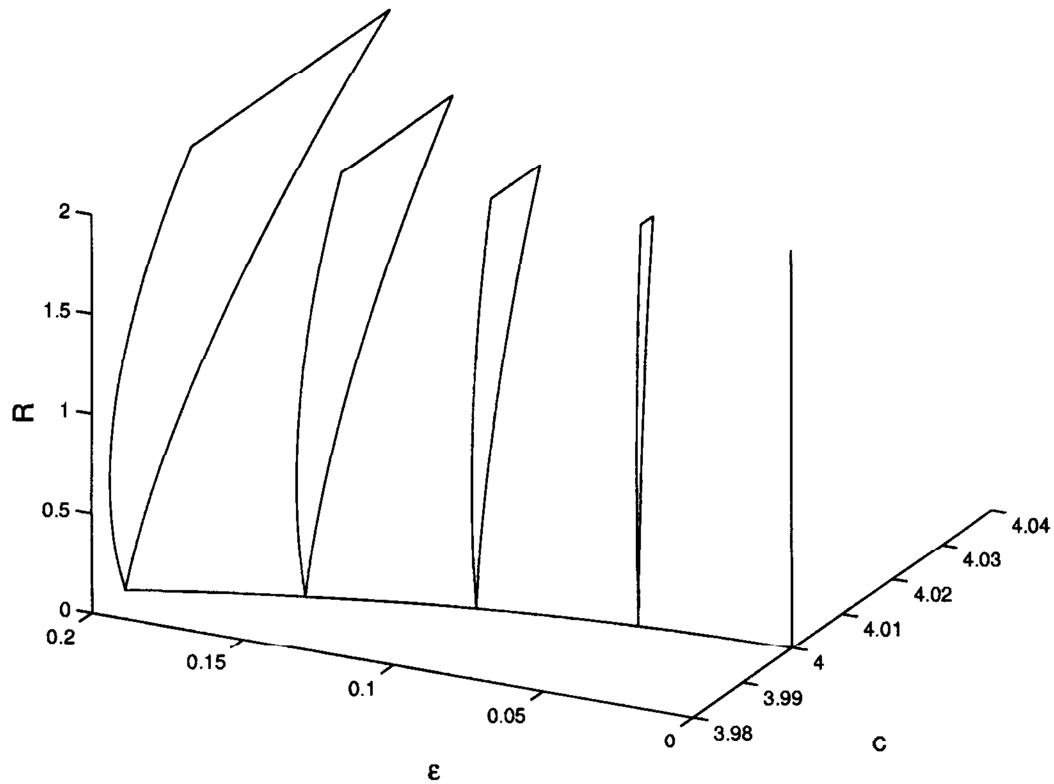


*Figure 11.* Bifurcating branches of slow flow equilibria. The line in the $c$-$\varepsilon$ plane corresponds to the coexistence curve $c = 4 - (1/3)\varepsilon^2$. The bifurcating branches of slow flow equilibria are shown for several slices in $c$ and $R$ corresponding to fixed values of $\varepsilon$.

coexisting solutions. However, we have just demonstrated that nonlinear terms can make the origin (nonlinearly) unstable.

## 6. Conclusions

We have found that nonlinear terms can affect the stability of the origin in parametrically excited systems which exhibit coexistence. A physical example of where this may occur is in a simplified two degree of freedom model for a thin elastica.

In the example we looked at, adding a quadratic nonlinearity to the system that exhibits coexistence makes the origin a degenerate equilibrium point in the slow-flow for parameter values where the linearization predicts coexisting solutions. For these parameter values, the origin is unstable, contrary to predictions made by linear theory.

In our bifurcation analysis of the slow-flow, we have found that for $c_2 > -49/120$ the system has a pair of nonorigin equilibria except in the special case when we are on the coexistence curve ($c_2 = -1/3$). Figure 11 shows another way to visualize the bifurcation diagram previously shown in Figure 4. The nonorigin equilibria in the slow-flow correspond to periodic motions in the original equation and periodic motions in the slow-flow correspond to quasi-periodic motions in the original equation.

In the original equation, the origin is stable and surrounded by a continuous family of quasi-periodic motions except on the coexistence curve ($c_2 = -1/3$) where the system is degenerate. The nonorigin center in the slow-flow corresponds to a stable periodic motion in the original equation. This stable periodic motion is created with a nonzero amplitude and continues to grow in amplitude as $c_2$ is increased. It is surrounded by a continuous family of quasi-periodic motions. The nonorigin saddle in the slow-flow corresponds to an unstable periodic motion in the original equation. The stable and unstable manifolds of this unstable periodic motion are homoclinic and separate regions of different continuous families of quasi-periodic motions. The exception is on the coexistence curve where the saddle and the origin coalesce.

Although a dynamical system which possesses a center is in general structurally unstable, we find that in this system the nonlinearity, $\varepsilon\alpha x^2$, is nondissipative, and so it is not able to cause instability without the degeneracy represented by coexistence. On the other hand, with this degeneracy it is not surprising that a nonlinear perturbation, even if nondissipative, is able to produce instability. However, although not discussed in this paper, we also looked at the effect of adding only a cubic nonlinearity of the form $\varepsilon\beta x^3$, but found the stability of the origin was not affected.

## Appendix A

We present details for the second order averaging calculation on Equation (41). Although the procedure is straight-forward, it is complicated algebraically and is best done using computer algebra software (MACSYMA).

We start by dividing Equation (41) by $1 - (\varepsilon/2)\cos 2t$ and Taylor expanding in $\varepsilon$ to get it in the form:

$$\ddot{x} + 4x = \varepsilon F_1(x, \dot{x}, t) + \varepsilon^2 F_2(x, \dot{x}, t) + O(\varepsilon^3). \tag{50}$$

Note that for $\varepsilon = 0$, Equation (50) has the solution:

$$x = \rho\cos(2t + \phi), \quad \dot{x} = -2\rho\sin(2t + \phi). \tag{51}$$

Using variation of parameters on $\rho$ and $\phi$, Equation (50) can be written as

$$\begin{aligned} \dot{\rho} &= -\frac{\varepsilon}{2}\sin(2t + \phi)F_1\left(\rho\cos(2t + \phi), -2\rho\sin(2t + \phi), t\right) \\ &\quad - \frac{\varepsilon^2}{2}\sin(2t + \phi)F_2\left(\rho\cos(2t + \phi), -2\rho\sin(2t + \phi), t\right) + O(\varepsilon^3), \end{aligned} \tag{52}$$

$$\begin{aligned} \dot{\phi} &= -\frac{\varepsilon}{2\rho}\cos(2t + \phi)F_1\left(\rho\cos(2t + \phi), -2\rho\sin(2t + \phi), t\right) \\ &\quad - \frac{\varepsilon^2}{2\rho}\cos(2t + \phi)F_2\left(\rho\cos(2t + \phi), -2\rho\sin(2t + \phi), t\right) + O(\varepsilon^3). \end{aligned} \tag{53}$$

We now use the near identity transformation:

$$\rho = R + \varepsilon w_1(R, \psi, t) + \varepsilon^2 v_1(R, \psi, t) + O(\varepsilon^3), \tag{54}$$

$$\phi = \psi + \varepsilon w_2(R, \psi, t) + \varepsilon^2 v_2(R, \psi, t) + O(\varepsilon^3). \tag{55}$$

Substituting Equations (54) and (55) into Equations (52) and (53), solving for $\dot{R}$ and $\dot{\psi}$ and Taylor expanding in $\varepsilon$ gives

$$\dot{R} = \varepsilon\left[-\frac{\partial w_1}{\partial t} - \frac{1}{2}\sin(2t+\psi)F_1\,(R\cos(2t+\psi), -2R\sin(2t+\psi), t)\right] + \varepsilon^2\left[-\frac{\partial v_1}{\partial t} + K_1(R,\psi,t)\right] + O(\varepsilon^3), \tag{56}$$

$$\dot{\psi} = \varepsilon\left[-\frac{\partial w_2}{\partial t} - \frac{1}{2R}\cos(2t+\psi)F_1\,(R\cos(2t+\psi), -2R\sin(2t+\psi), t)\right] + \varepsilon^2\left[-\frac{\partial v_2}{\partial t} + K_2(R,\psi,t)\right] + O(\varepsilon^3), \tag{57}$$

where $K_1$ and $K_2$ depend on $w_1$, $w_2$, $F_1$ and $F_2$.

In first order averaging, $\partial w_1/\partial t$ and $\partial w_2 \partial t$ are chosen to simplify $O(\varepsilon)$ terms as much as possible. The usual approach is to trigonometrically reduce the equations and choose $w_1$ and $w_2$ to remove all the trigonometric terms in $t$. The result of first order averaging is a pair of equations of the form

$$\dot{R} = \varepsilon G_{11}(R,\psi) + O(\varepsilon^2), \tag{58}$$

$$\dot{\psi} = \varepsilon G_{21}(R,\psi) + O(\varepsilon^2). \tag{59}$$

Once $w_1$ and $w_2$ have been obtained, we can go to second order where $\partial v_1/\partial t$ and $\partial v_2/\partial t$ are chosen to simplify $O(\varepsilon^2)$ terms as much as possible. Second order averaging results in a pair of equations of the form:

$$\dot{R} = \varepsilon G_{11}(R,\psi) + \varepsilon^2 G_{12}(R,\psi) + O(\varepsilon^3), \tag{60}$$

$$\dot{\psi} = \varepsilon G_{21}(R,\psi) + \varepsilon^2 G_{22}(R,\psi) + O(\varepsilon^3). \tag{61}$$

The resulting differential equations, Equations (60) and (61), are known as the slow-flow equations.

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