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NONLINEAR EFFECTS ON COEXISTENCE PHENOMENON IN PARAMETRIC EXCITATION

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ABSTRACT

We investigate the effect of nonlinearites on a parametrically excited ordinary differential equation whose linearization exhibits the phenomena of coexistence. The differential equation studied governs the stability of a mode of vibration in an unforced conservative two degree of freedom system used to model the free vibrations of a thin elastica. Using perturbation methods, we show that at parameter values corresponding to coexistence, nonlinear terms can cause the origin to become nonlinearly unstable, even though linear stability analysis predicts the origin to be stable. We also investigate the bifurcations associated with this instability.

INTRODUCTION

In this work we look at the following parametrically excited ordinary differential equation (ODE):

$$\left(1-\frac{\varepsilon}{2}\cos 2t\right)\ddot{x}+\varepsilon\sin 2t\,\dot{x}+cx+\varepsilon\alpha x^2=0 \tag{1}$$

When $\alpha = 0$, Eq.(1) arises in the study of the dynamics of a thin elastica which was the subject of the Ph.D thesis of Cusumano [1]. Also when $\alpha = 0$, Eq.(1) is a form of Ince's equation and exhibits the phenomena of coexistence [3]. By taking $\alpha > 0$, we add a nonlinear spring to the physical model previously studied in [1],[4], permitting us to investigate the effect of nonlinearities on a system exhibiting coexistence.

We start by reviewing the phenomena of coexistence in Ince's equation. Next, we derive Eq.(1) from a model proposed by Cusumano [1] and show how the quadratic nonlinearity may enter in the equation. Finally, we investigate the effect of the quadratic nonlinearity in Eq.(1) by using pertubation methods.

REVIEW OF COEXISTENCE AND INCE'S EQUATION

A well-known ordinary differential equation with periodic coefficients is Mathieu's equation:

$$\ddot{z}+(\delta+\varepsilon\cos 2t)z=0 \tag{2}$$

For given values of the parameters δ and ε, either all the solutions of are bounded (*stable*) or an unbounded solution exists (*unstable*). The curves separating the stable and unstable regions in the δ-ε plane are known as *transition curves* (see Fig.1). The instability intervals which emanate out of the δ axis at values $\delta = n^2$ for $n = 1,2,3,\ldots$ are commonly referred to as *resonance tongues* .

The presence of resonance tongues is a generic feature of differential equations with periodic coefficients. One phenomena that does *not* occur in Mathieu's equation but which may occur in other differential equations with periodic coefficients is *coexistence* . The phenomena of coexistence involves the disappearance of resonance tongues that would normally be present. In systems that exhibit coexistence, the two transitions curves that would normally define a resonance tongue coincide and the tongue closes up. An example of an equation that may exhibit coexistence is Ince's equation [3]:

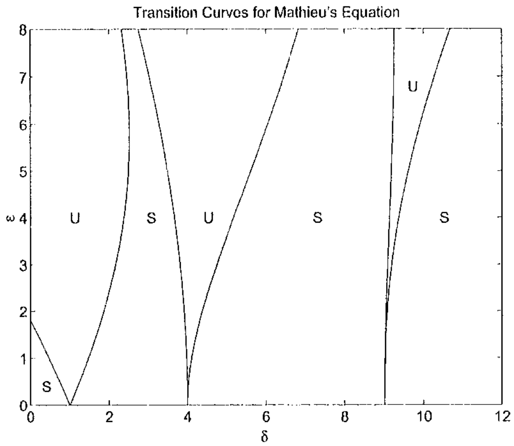

Figure 1. Transition curves separating regions of stability from instability in Mathieu's equation.

$$(1+a\cos 2t)\ddot{x}+b\sin 2t\,\dot{x}+(c+d\cos 2t)x=0 \tag{3}$$

Note that Mathieu's equation is a special case of Ince's equation which turns out not to exhibit coexistence. We know from Floquet theory that since the periodic coefficients in Eq.(3) have period π, the solutions on the transition curves will be periodic with period π or 2π. Let us assume that a, b and d depend on ε. If $a=b=d=0$ when $\varepsilon=0$, then Eq.(3) becomes:

$$\ddot{x}+cx=0 \tag{4}$$

which has solutions of period $\frac{2\pi}{\sqrt{c}}$. These correspond to solutions of period π or 2π when $\frac{2\pi}{\sqrt{c}}=\frac{2\pi}{n}$ so we would expect resonance tongues in the c-ε plane to emanate from the points $c=n^2$, $n=1,2,3,\ldots$ on the c-axis.

We use the method of harmonic balance to investigate what happens to the tongues of instability in Ince's equation [3],[7]. Since the transition curves are characterized by having solutions of period π or 2π, we expand the solution x in a Fourier series:

$$x(t)=\sum_{n=0}^{\infty} a_n\cos nt+b_n\sin nt \tag{5}$$

Substituting Eq.(5) into Eq.(3), simplifying and collecting trigonometric terms, we obtain four sets of algebraic equations on the coefficients a_n and b_n. Each set deals exclusively with a_{even}, b_{even}, a_{odd} and b_{odd}. Also, each set is homogeneous and of infinite order so for a nontrivial solution, the determinants must vanish. The four infinite determinants are:

$$a_{even}:\left|\begin{array}{ccccccc} c & \frac{d}{2}-b-2a & 0 & 0 & 0 & \\ d & c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ 0 & \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \cdots \\ 0 & 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \\ 0 & 0 & 0 & \frac{d}{2}+3b-18a & c-64 & \\ & & & \cdots & & \end{array}\right| \tag{6}$$

$$b_{even}:\left|\begin{array}{ccccc} c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \cdots \\ 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \\ 0 & 0 & \frac{d}{2}+3b-18a & c-64 & \\ & & \cdots & & \end{array}\right| \tag{7}$$

$$a_{odd}:\left|\begin{array}{ccccc} c-1+\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \cdots \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{array}\right| \tag{8}$$

$$b_{odd}:\left|\begin{array}{ccccc} c-1-\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \cdots \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{array}\right| \tag{9}$$

The notation in these determinants can be simplified by setting (after Magnus and Winkler, "Hill's Equation" [3]):

$$Q(m)=\frac{d}{2}+bm-2am^2 \tag{10}$$

$$P(m)=Q\left(m-\frac{1}{2}\right)=\frac{d+b(2m-1)-a(2m-1)^2}{2} \tag{11}$$

Using this notation for $Q(m)$ and $P(m)$, taking the four infinite determinants in Eqs.(6-9) and setting them to zero gives:

$$a_{even}:\left|\begin{array}{cccccc} c & Q(-1) & 0 & 0 & 0 & \\ 2Q(0) & c-4 & Q(-2) & 0 & 0 & \\ 0 & Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & 0 & Q(2) & c-36 & Q(-4) & \\ 0 & 0 & 0 & Q(3) & c-64 & \\ & & & \cdots & & \end{array}\right|=0 \tag{12}$$

$$b_{even}:\left|\begin{array}{ccccc} c-4 & Q(-2) & 0 & 0 & \\ Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & Q(2) & c-36 & Q(-4) & \\ 0 & 0 & Q(3) & c-64 & \\ & & \cdots & & \end{array}\right|=0 \tag{13}$$

$$a_{odd}:\left|\begin{array}{ccccc} c-1+P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(3) & c-25 & P(-3) & \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{array}\right|=0 \tag{14}$$

$$b_{odd}:\left|\begin{array}{ccccc} c-1-P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \cdots \\ 0 & P(2) & c-25 & P(-3) & \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{array}\right|=0 \tag{15}$$

Comparison of the determinants in Eq.(12) and Eq.(13) shows that if the first row and first column of Eq.(12) are removed, then the remainder of Eq.(12) is identical to Eq.(13). The significance of this observation is that if any one of the off-diagonal terms vanishes, that is if $Q(m) = 0$ for some integer m (positive, negative or zero), then coexistence can occur and an infinite number of possible tongues of instability will not occur.

In order to understand how this works, suppose $Q(2) = 0$. Then we may represent Eqs.(12),(13) symbolically as follows:

$$a_{even}: \left| \begin{array}{cccccc} X & X & 0 & 0 & 0 & \\ X & X & X & 0 & 0 & \\ 0 & X & X & X & 0 & \cdots \\ 0 & 0 & Q(2) & X & X & \\ 0 & 0 & 0 & X & X & \\ & & & \cdots & & \end{array} \right| = 0 \quad (16)$$

$$b_{even}: \left| \begin{array}{ccccc} X & X & 0 & 0 & \\ X & X & X & 0 & \cdots \\ 0 & Q(2) & X & X & \\ 0 & 0 & X & X & \\ & & \cdots & & \end{array} \right| = 0 \quad (17)$$

where we have used the symbol X to represent a term which is non-zero. The vanishing of $Q(2)$ "disconnects" the lower (infinite) portion of these equations from the upper (finite) portion. There are now two possible ways in which to satisfy these equations with $Q(2) = 0$.

1. For a nontrivial solution to the lower (infinite) portion, the (disconnected, infinite) determinant must vanish. Since this determinant is identical for both the a's and b's, coexistence is present and the associated tongues do not occur. In this case the upper portion of the determinant will not vanish in general, and the coefficients a_0, a_2, a_4, b_2 and b_4 will not be zero.

2. Another possibility is that the infinite determinant of the lower portion is not zero, requiring that the associated a_{even} and b_{even} coefficients vanish. With these a's and b's zero, the upper portion of the system becomes independent of the lower, and for a nontrivial solution for a_0, a_2,a_4, b_2 and b_4, the upper portion of both determinants must vanish. For Eq.(16) this involves a 3×3 determinant and yields a cubic on c, while for Eq.(17) this involves a 2×2 determinant and gives a quadratic on c. Together these yield 5 expressions for c in terms of the other parameters of the problem, which, if real, correspond to the 5 transition curves. One of these passes through the c-axis at $c = 0$, and the other 4 produce tongues of instability emanating from $c = 4$ and $c = 16$ respectively.

A similar situation occurs for Eqs.(14),(15). If $P(m) = 0$ for some integer m (positive, negative or zero) then only a finite number of tongues will occur from amongst the infinite set of tongues which emanate from the points $c = (2n-1)^2$, $n = 1, 2, 3, \ldots$ on the c-axis.

As an example we take Eq.(1) when $\alpha = 0$. It is in the form of Ince's equation with the parameter values:

$$a = -\frac{\varepsilon}{2}, \quad b = \varepsilon, \quad d = 0 \quad (18)$$

For the parameter values in Eqs.(18), the polynomials $Q(m)$ and $P(m)$ from Eqs.(19),(20) become:

$$Q(m) = \varepsilon m + \varepsilon m^2 \quad (19)$$

$$P(m) = \frac{2\varepsilon(2m-1) + \varepsilon(2m-1)^2}{4} \quad (20)$$

For Eqs.(19),(20), it is easy to show that $Q(m) = 0$ when $m = 0, -1$ and $P(m) = 0$ when $m = \pm 1/2$. Substituting $Q(0) = 0$ and $Q(-1) = 0$ into Eq.(12), we see that the element c in the upper left corner of Eq.(12) becomes disconnected from the rest of the infinite determinant, which is itself identical to the infinite determinant of Eq.(13). From this we can conclude that $c = 0$ is a transition curves and all the even tongues disappear. Because $P(m)$ does not have integer roots, we can also conclude that the system has an infinite number of odd tongues.

Some other examples of systems exhibiting coexistence are given in [9], [5], [2], [10] and [7].

DERIVATION OF DIFFERENTIAL EQUATION

In this section we derive Eq.(1) from a model proposed by Cusumano [1]. In his thesis, Cusumano [1] studied the dynamics of a thin elastica. He showed that complicated dynamics result and that a mode of vibration exists which involves both bending and torsional modes. Fig.2 shows some of the modes of vibration of a thin elastica. To get a better understanding of the dynamics, Cusumano [1] examined the simplified two degree of freedom model shown in Fig.3.

In the simplified model, the rotational motion due to coordinate q_1 is associated with the torsional motion of the elastica and the rectilinear deflection due to q_2 is associated with the bending motion. Pak et al [4] investigated the different modes of vibration for the system in Fig.3. They found that the stability of the bending mode is governed by an equation of the form of Eq.(1) with $\alpha = 0$.

We consider a system similar to the one shown in Fig.3 except with a nonlinear torsional spring. Instead of using a linear force-displacement relation as in Fig.3:

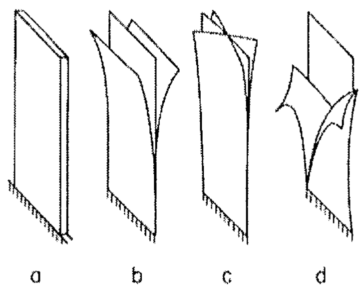

Figure 2. Thin elastica. (a) undeformed; (b) bending mode; (c) torsional mode; (d) non-local mode, involving both bending and torsion. (Pak et al [4])

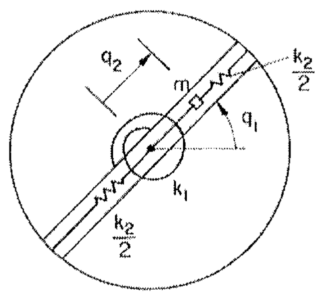

Figure 3. Simplified two degree of freedom model of an elastica (Cusumano [1])

$$f = k_1 q_1 \quad (21)$$

We use the following nonlinear force-displacement relation with an additional quadratic term:

$$f = k_{11} q_1 + k_{12} q_1^2 \quad (22)$$

If we set $k_{12} = 0$ and $k_{11} = k_1$, we get back the system in Fig.3. Note that the result of adding the quadratic term is to add some asymmetry in the torsional spring. Much of the derivation presented here follows the paper by Pak et al [4].
We begin by writing the kinetic and potential energies for the system:

$$T = \frac{m}{2}\left[l^2 (q_1')^2 + (q_2')^2\right] + \frac{J}{2}(q_1')^2 \quad (23)$$

$$V = \frac{1}{2}\left[k_{11} q_1^2 + \frac{2}{3} k_{12} q_1^3 + k_2 q_2^2\right] \quad (24)$$

Using the rescalings:

$$x = \sqrt{\frac{Jk_2}{m}}\, q_1, \quad y = \sqrt{k_2}\, q_2, \quad t = \sqrt{\frac{k_2}{m}}\, \tau \quad (25)$$

the following Lagrangian L can be obtained:

$$L = \frac{1}{2}(1 + \gamma y^2)\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\left(\kappa_1 x^2 + \frac{2}{3}\kappa_2 x^3 + y^2\right) \quad (26)$$

where,

$$\gamma = \frac{m}{Jk_2}, \quad \kappa_1 = \frac{k_{11}/J}{k_2/m}, \quad \kappa_2 = k_{12}\left(\frac{m}{Jk_2}\right)^{3/2} \quad (27)$$

Applying Lagrange's equations to Eq.(26) gives the equations of motion:

$$(1 + \gamma y^2)\ddot{x} + 2\gamma y \dot{y} \dot{x} + \kappa_1 x + \kappa_2 x^2 = 0 \quad (28)$$
$$\ddot{y} - \gamma \dot{x}^2 y + y = 0 \quad (29)$$

Note that the x-mode, $y \equiv 0$, and the y-mode, $x \equiv 0$, are exact solutions to Eqs.(28),(29). To investigate the stability of the y-mode, $x \equiv 0$, we linearize Eqs.(28),(29) about the exact solution by setting:

$$x = 0 + \mu \hat{x}, \quad y = A \sin t + \mu^{3/2}\, \hat{y} \quad (30)$$

where μ is a small parameter. The factor $\mu^{3/2}$ in Eqs.(30) is chosen so that the scalings come out appropriately. Substituting Eqs.(30) into Eqs.(28),(29) and Taylor expanding in μ gives:

$$\mu(1 + \gamma A^2 \sin^2 t)\ddot{\hat{x}} + \mu\gamma A^2 \sin 2t\, \dot{\hat{x}} + \mu\kappa_1 \hat{x} + \mu^2 \kappa_2 \hat{x}^2 + O(\mu^{5/2}) = 0 \quad (31)$$
$$\mu^{3/2}\, \ddot{\hat{y}} + \mu^{3/2}\, \hat{y} + O(\mu^2) = 0 \quad (32)$$

Note that Eq.(31) is uncoupled from $\hat{y}$ up to $O(\mu^2)$. Taking terms up to $O(\mu^2)$ in Eq.(31), setting $\mu = 1$ and dropping hats gives:

$$(1+\gamma A^2\sin^2 t)\ddot{x}+\gamma A^2\sin 2t\,\dot{x}+\kappa_1 x+\kappa_2 x^2=0 \tag{33}$$

Expanding the trigonometric term in Eq.(33) gives:

$$\left(1+\frac{\gamma A^2}{2}-\frac{\gamma A^2}{2}\cos 2t\right)\ddot{x}+\gamma A^2\sin 2t\dot{x}+\kappa_1 x+\kappa_2 x^2=0 \tag{34}$$

Finally, dividing Eq.(34) by $1+\frac{\gamma A^2}{2}$ and taking:

$$\varepsilon=\frac{\gamma A^2}{1+\frac{\gamma A^2}{2}},\quad c=\frac{\kappa_1}{1+\frac{\gamma A^2}{2}},\quad \varepsilon\alpha=\frac{\kappa_2}{1+\frac{\gamma A^2}{2}} \tag{35}$$

we have obtained Eq.(1):

$$\left(1-\frac{\varepsilon}{2}\cos 2t\right)\ddot{x}+\varepsilon\sin 2t\,\dot{x}+cx+\varepsilon\alpha x^2=0$$

DERIVATION OF COEXISTENCE CURVE

We obtain a series expansion for small ε of the curve of coexisting solutions emanating out of $c=4$ in Eq.(1) when $\alpha=0$. The system exhibits coexistence for the even resonance tongues so the curves of coexisting solutions can be computed from the infinite determinant for b_{even} (Eq.(7)). We start by taking a finite truncation of the infinite determinant for b_{even}. If we take a 4×4 truncation we get:

$$b_{even}:\begin{vmatrix} c-4 & \frac{d}{2}-2b-8a & 0 & 0 \\ \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 \\ 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a \\ 0 & 0 & \frac{d}{2}+3b-18a & c-64 \end{vmatrix}=0 \tag{36}$$

It is a straight-forward computation to obtain a single equation for the determinant of Eq.(36). To get an approximation for the curve of coexisting solutions out of $c=4$, we Taylor expand c in a powers series in ε about $c=4$:

$$c=4+\varepsilon c_1+\varepsilon^2 c_2+\varepsilon^3 c_3+\ldots \tag{37}$$

Recall for Eq.(1), the parameter values are:

$$a=-\frac{\varepsilon}{2},\quad b=\varepsilon,\quad d=0$$

Substituting these values for a, b, c and d into the single equation obtained from Eq.(36) and Taylor expanding in ε we get:

$$\begin{aligned}-23040c_1\varepsilon+(-23040c_2+3024c_1^2-7680)\varepsilon^2 \\ +(-23040c_3+6048c_1c_2-104c_1^3+4256c_1)\varepsilon^3+\ldots=0\end{aligned} \tag{38}$$

Requiring terms of $O(\varepsilon)$ in Eq.(38) to be zero gives $c_1=0$. Using this value for c_1 and requiring terms of $O(\varepsilon^2)$ to be zero gives $c_2=-1/3$. Using these values for c_1 and c_2, higher order terms can also be obtained in a similar fashion. Of course, obtaining higher order terms also requires taking more terms in the truncation of the determinant for b_{even}. To $O(\varepsilon^2)$, the series expansion for the curve of coexisting solutions emanating out of $c=4$ is:

$$c=4-\frac{1}{3}\varepsilon^2+\ldots \tag{39}$$

EFFECT OF A QUADRATIC NONLINEARITY

We use the method of averaging to investigate the effect of the quadratic nonlinearity in Eq.(1). For more on the method of averaging see Rand [8], [6]. First, note that α can be rescaled out of Eq.(1). We assume $\alpha>0$, so without loss of generality we set $\alpha=1$. From our results for $\alpha=0$, we know that the Eq.(1) exhibits coexistence and the tongues of instability that emanate out of $c=n^2$ for n even vanish. To perturb off a resonance where coexistence occurs, we set:

$$c=4+\varepsilon^2 c_2 \tag{40}$$

Substituting Eq.(40) and $\alpha=1$ into Eq.(1) we get:

$$\left(1-\frac{\varepsilon}{2}\cos 2t\right)\ddot{x}+\varepsilon\sin 2t\,\dot{x}+(4+\varepsilon^2 c_2)x+\varepsilon x^2=0 \tag{41}$$

For small values of ε, we can apply the method of averaging to Eq.(41). See Appendix A for details of the averaging calculation. To first order, the averaging procedure does not produce any terms in the slow-flow equations. Going to second order in the averaging calculation results in the following slow-flow equations:

$$\dot{R} = \frac{\varepsilon^2 R^2 \sin\psi}{48} \tag{42}$$

$$\dot{\psi} = \frac{\varepsilon^2\left(24c_2 - 5R^2 + 6R\cos\psi + 8\right)}{96} \tag{43}$$

We start by looking for equilibria in the slow-flow. Note that $R = 0$ is always an equilibrium point. Setting the RHS of Eq.(42) to zero requires $\sin\psi = 0$ so $\psi = 0$ or $\psi = \pi$. Substituting $\psi = 0$ and $\psi = \pi$ into the RHS of Eq.(43) and setting it to zero gives:

$$24c_2 - 5R^2 + 6R + 8 = 0 \tag{44}$$

$$24c_2 - 5R^2 - 6R + 8 = 0 \tag{45}$$

Eqs.(44),(45) differ only by the sign of R term and both have the same discriminant. For real roots, we require the discriminant to be positive which results in:

$$36 + 20(24c_2 + 8) \geq 0 \tag{46}$$

This gives the condition $c_2 \geq -49/120$. When $c_2 = -49/120$, a pair of equilbria are created at $R = 3/5$, $\psi = 0$. Also note that for $R = 0$ both Eqs.(44),(45) are satisfied when $c_2 = -1/3$. This value of c_2 corresponds with the perturbation expansion for the curve of coexisting solutions emanating from $c = 4$ when $\alpha = 0$ (see Eq.(39)). Fig.4 shows a bifurcation diagram of the slow-flow equilibria by plotting Eqs.(44),(45).

For $c_2 < -49/120$, the origin is the only equilbrium point. A pair of equilibria are created when $c_2 = -49/120$ and for $-49/120 < c_2 < -1/3$ there are two nontrivial equilibria with $\psi = 0$. At $c_2 = -1/3$, one of the nontrivial equilibria goes through the origin (recall that the origin is always an equilibrium point) and ψ for that equilibria changes from $\psi = 0$ to $\psi = \pi$. For $c_2 > -1/3$, there is one nontrivial equilibria with $\psi = 0$ and one with $\psi = \pi$.

To investigate the nature of the bifurcations, it is more convenient to look at the slow-flow equations in cartesian coordinates. Transforming to cartesian coordinates $u = R\cos\psi$, $v = -R\sin\psi$, Eqs.(42),(43) become:

$$\dot{u} = \frac{\varepsilon^2 v\left[24c_2 + 8 + 4u - 5(u^2 + v^2)\right]}{96} \tag{47}$$

$$\dot{v} = \frac{\varepsilon^2\left[5u(u^2 + v^2) - 6(u^2 + v^2) + 4v^2 - 24c_2 u - 8u\right]}{96} \tag{48}$$

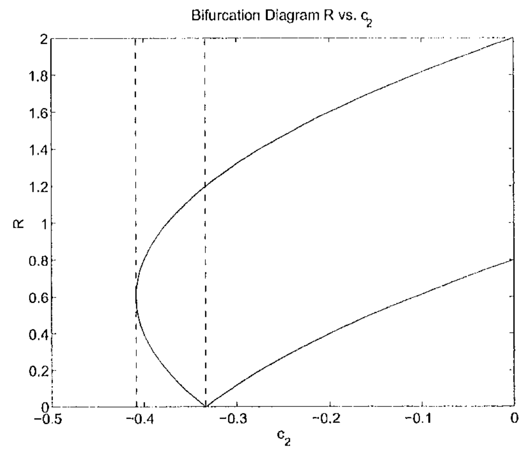

Figure 4. Bifurcation diagram of slow-flow equilibria. Dotted lines correspond to $c_2 = -1/3$ and $c_2 = -49/120$. Here c_2 is related to c by $c = 4 + c_2\varepsilon^2$. $c_2 = -1/3$ corresponds to coexistence.

Eqs.(47),(48) have the first integral:

$$\frac{5v^4 + (10u^2 - 8u - 48c_2 - 16)v^2 + 5u^4 - 8u^3 - (48c_2 + 16)u^2}{4} = K \tag{49}$$

Using Eq.(49), we can investigate the stability of the equilibria by plotting invariant curves of the system. Figs.5-9 show the sequence of invariant curves for the system as c_2 is increased. Figs.5,7,9 are representative of the system in the regions $c_2 < -49/120$, $-49/120 < c_2 < -1/3$ and $c_2 > -1/3$ respectively. Figs.6,8 correspond to parameter values where bifurcations occur ($c_2 = -49/120$ and $c_2 = -1/3$).

For $c_2 < -49/120$, the only equilibria is the origin which is a center. At $c_2 = -49/120$, a saddle-center bifurcation occurs where a saddle and a center are created. As c_2 approaches the value $c_2 = -1/3$, the region of stability around the origin gets smaller as the saddle created in the saddle-center bifurcation moves toward the center at the origin. At the critical value $c_2 = -1/3$, the saddle coalesces with the origin and this equilibrium point at the origin is degenerate and unstable. Fig.10 shows a blow-up of the invariant curves around the origin when $c_2 = -1/3$.

As c_2 increases and goes through $c_2 = -1/3$, the saddle moves through the origin and the saddle and center created in the saddle-center bifurcation are now on opposite sides of the origin which remains a center. As c_2 further increases, the nontrivial equilibria move farther away from the origin.

As c_2 approaches the value $c_2 = -1/3$ from either side, the region of stability around the origin gets smaller as a saddle and

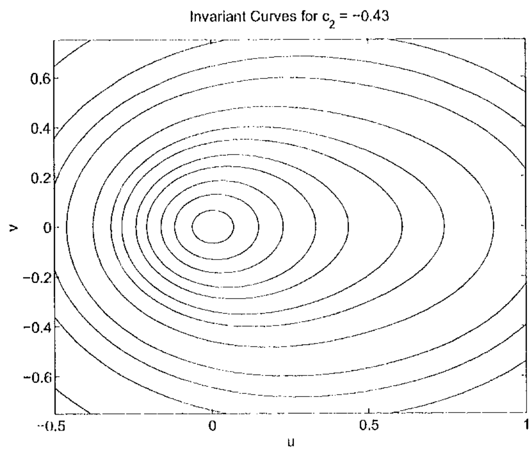

Figure 5. Invariant curves for $c_2 = -0.43$

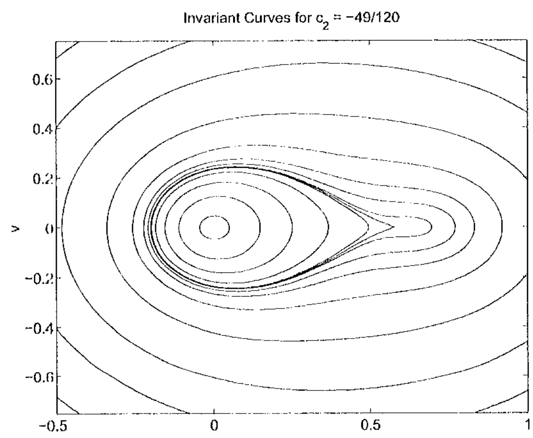

Figure 6. Invariant curves for $c_2 = -49/120$. Note the saddle-center bifurcation at $x = 3/5$, $y = 0$.

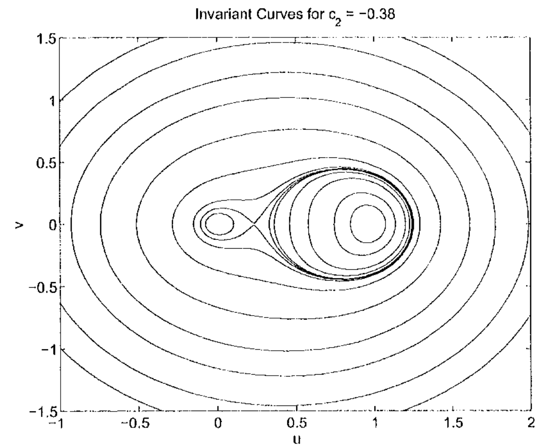

Figure 7. Invariant curves for $c_2 = -0.38$

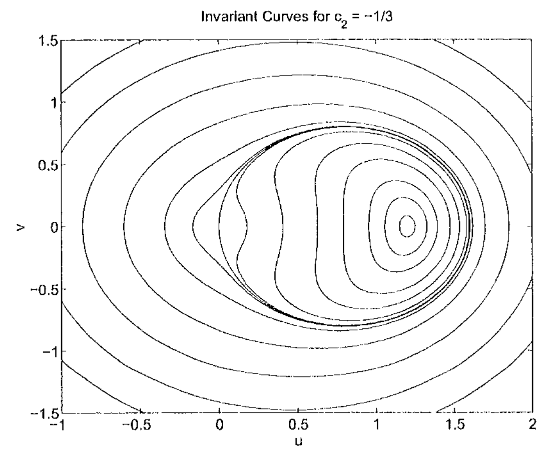

Figure 8. Invariant curves for $c_2 = -1/3$. Note the origin is a degenerate equilibrium point.

center come together. The value $c_2 = -1/3$ corresponds to coexistence. Linear stability analysis predicts that the origin is stable along the curve of coexisting solutions. However, we have just demonstrated that nonlinear terms can make the origin (nonlinearly) unstable.

CONCLUSIONS

We have found that nonlinear terms can affect the stability of the origin in parametrically excited systems which exhibit coexistence. A physical example of where this may occur is in a simplified two degree of freedom model for a thin elastica.

In the example we looked at, adding a quadratic nonlinearity to the system that exhibits coexistence makes the origin a degenerate equilibrium point in the slow-flow for parameter values where the linearization predicts coexisting solutions. For these parameter values, the origin is unstable, contrary to predictions made by linear theory.

In our bifurcation analysis of the slow-flow, we have found that for $c_2 > -49/120$ the system has a pair of non-trivial equilibria except in the special case when we are on the coexistence curve ($c_2 = -1/3$). The non-trivial equilibria in the slow-flow correspond to periodic motions in the original equation and periodic motions in the slow-flow correspond to quasi-periodic motions in the original equation. Note that the origin in the slow-flow is still the origin in the original equation (although we could

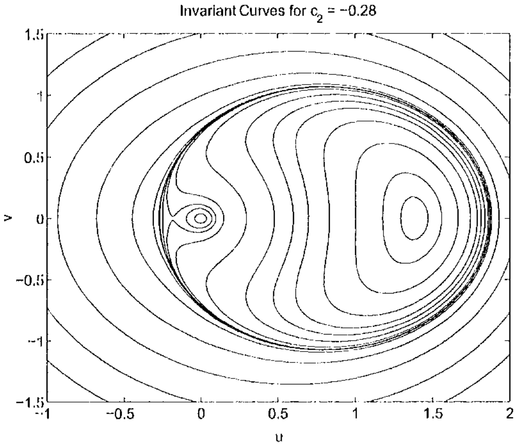

Figure 9. Invariant curves for $c_2 = -0.28$

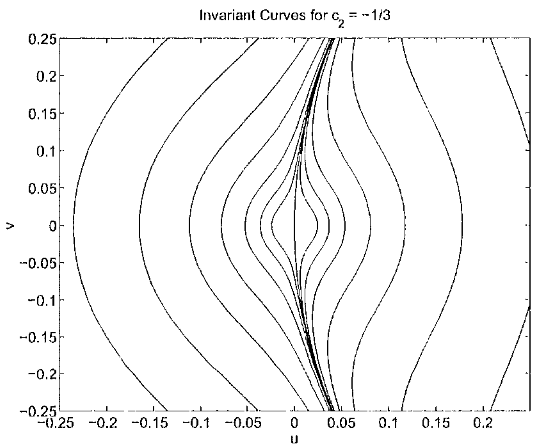

Figure 10. Invariant curves for $c_2 = -1/3$ near the origin

also think of the origin as a periodic motion because the original equation is non-autonomous).

In the original equation, the origin is stable and surrounded by a continuous family of quasi-periodic motions except on the coexistence curve ($c_2 = -1/3$) where the system is degenerate. The non-trivial center in the slow-flow corresponds to a stable periodic motion in the original equation. This stable periodic motion is created with a non-zero amplitude and continues to grow in amplitude as c_2 is increased. The stable periodic motion is also surrounded by a continuous family of quasi-periodic motions. The non-trivial saddle in the slow-flow corresponds to an unstable periodic motion in the original equation. The stable and unstable manifolds of this unstable periodic motion separate regions of different continuous families of quasi-periodic motions. The exception is on the coexistence curve where the saddle and the origin coalesce.

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Appendix A

We present details for the second order averaging calculation on Eq.(41). Although the procedure is straight-forward, it is complicated algebraically and is best done using computer algebra software (MACSYMA).

We start by dividing Eq.(41) by $1-\frac{\varepsilon}{2}\cos 2t$ and Taylor expanding in ε to get it in the form:

$$\ddot{x}+4x=\varepsilon F_1(x,\dot{x},t)+\varepsilon^2 F_2(x,\dot{x},t)+O(\varepsilon^3) \tag{50}$$

Note that for ε=0, Eq.(50) has the solution:

$$x=\rho\cos(2t+\phi), \quad \dot{x}=-2\rho\sin(2t+\phi) \tag{51}$$

Using variation of parameters on ρ and ϕ, Eq.(50) can be written as:

$$\begin{aligned}\dot{\rho} = &-\frac{\varepsilon}{2}\sin(2t+\phi)F_1(\rho\cos(2t+\phi),-2\rho\sin(2t+\phi),t) \qquad (52)\\ &-\frac{\varepsilon^2}{2}\sin(2t+\phi)F_2(\rho\cos(2t+\phi),-2\rho\sin(2t+\phi),t)+O(\varepsilon^3)\\ \dot{\phi} = &-\frac{\varepsilon}{2\rho}\cos(2t+\phi)F_1(\rho\cos(2t+\phi),-2\rho\sin(2t+\phi),t) \qquad (53)\\ &-\frac{\varepsilon^2}{2\rho}\cos(2t+\phi)F_2(\rho\cos(2t+\phi),-2\rho\sin(2t+\phi),t)+O(\varepsilon^3)\end{aligned}$$

We now use the near identity transformation:

$$\rho = R+\varepsilon w_1(R,\psi,t)+\varepsilon^2 v_1(R,\psi,t)+O(\varepsilon^3) \tag{54}$$
$$\phi = \psi+\varepsilon w_2(R,\psi,t)+\varepsilon^2 v_2(R,\psi,t)+O(\varepsilon^3) \tag{55}$$

Substituting Eqs.(54),(55) into Eqs.(52),(53), solving for $\dot{R}$ and ψ and Taylor expanding in ε gives:

$$\begin{aligned}\dot{R} = &\ \varepsilon\left[-\frac{\partial w_1}{\partial t}-\frac{1}{2}\sin(2t+\psi)F_1(R\cos(2t+\psi),-2R\sin(2t+\psi),t)\right] \qquad (56)\\ &+\varepsilon^2\left[-\frac{\partial v_1}{\partial t}+K_1(R,\psi,t)\right]+O(\varepsilon^3)\\ \dot{\psi} = &\ \varepsilon\left[-\frac{\partial w_2}{\partial t}-\frac{1}{2R}\cos(2t+\psi)F_1(R\cos(2t+\psi),-2R\sin(2t+\psi),t)\right] \qquad (57)\\ &+\varepsilon^2\left[-\frac{\partial v_2}{\partial t}+K_2(R,\psi,t)\right]+O(\varepsilon^3)\end{aligned}$$

where, K_1 and K_2 depend on w_1,w_2,F_1 and F_2.

In first order averaging, $\frac{\partial w_1}{\partial t}$ and $\frac{\partial w_2}{\partial t}$ are chosen to simplify $O(\varepsilon)$ terms as much as possible. The usual approach is to trigonometrically reduce the equations and choose w_1 and w_2 to remove all the trigonometric terms in t. The result of first order averaging is a pair of equations of the form:

$$\dot{R} = \varepsilon G_{11}(R,\psi)+O(\varepsilon^2) \tag{58}$$
$$\dot{\psi} = \varepsilon G_{21}(R,\psi)+O(\varepsilon^2) \tag{59}$$

Once w_1 and w_2 have been obtained, we can go to second order where $\frac{\partial v_1}{\partial t}$ and $\frac{\partial v_2}{\partial t}$ are chosen to simplify $O(\varepsilon^2)$ terms as much as possible. Second order averaging results in a pair of equations of the form:

$$\dot{R} = \varepsilon G_{11}(R,\psi)+\varepsilon^2 G_{12}(R,\psi)+O(\varepsilon^3) \tag{60}$$
$$\dot{\psi} = \varepsilon G_{21}(R,\psi)+\varepsilon^2 G_{22}(R,\psi)+O(\varepsilon^3) \tag{61}$$

The resulting differential equations, Eqs.(60),(61), are known as the slow-flow equations.