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COUPLED OSCILLATORS AS A MODEL FOR NONLINEAR PARAMETRIC EXCITATION

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Introduction

In this paper the behavior of three conservative systems exhibiting parametric excitation are compared: the linear Mathieu equation, a nonlinear Mathieu equation and an autonomous system of two coupled nonlinear oscillators. The comparison is based on approximate Poincaré maps obtained by using perturbation methods involving near identity canonical transformations.

It is well known [1] that the zero solution of the linear Mathieu equation

$$\ddot{x} + (\delta+\varepsilon\cos t)x = 0 \tag{1}$$

exhibits instability for small values of ε when $\delta \simeq n^2/4$, $n = 1,2,3,\ldots$. The greatest instability occurs for $n = 1$ in which case the driver, $\cos t$, has twice the frequency of the unforced $(\varepsilon=0)$ system. For values of δ, ε for which $x(t) \equiv 0$ is unstable, eq. (1) predicts that motions starting near zero eventually become unbounded.

When a nonlinear term is added to eq. (1),

$$\ddot{x} + (\delta+\varepsilon\cos t)x + \varepsilon\alpha x^3 = 0 \tag{2}$$

it has been shown [2], [3] that when the zero solution is unstable, motions starting near zero move away from zero but do not become unbounded. The nonlinearity gives rise to a shift in frequency of the unforced system which may be thought of as balancing the parametric resonance [3].

It is the purpose of this work to compare systems (1) and (2) with the following system of two coupled oscillators:

$$\ddot{x} + (\delta+\varepsilon y)x + \varepsilon\alpha x^3 = 0 \tag{3.1}$$

$$\ddot{y} + y = -\varepsilon x^2/2 \tag{3.2}$$

Eq. (3.1) is analogous to eq. (2) if we identify $y(t)$ with the forcing function $\cos t$ of (2). Indeed from eq. (3.2), $y(t) = \cos t$ when the coupling term on the right hand side of (3.2) is zero. We call $x(t)$ the response variable and $y(t)$ the motor variable. This system represents parametric excitation with feedback and models situations in which the motor characteristics are dependent upon its load.

Abbreviated Communication

Analysis

In order to obtain approximate descriptions of the behavior of systems (1) - (3), we will utilize their conservative nature. Each has the following Hamiltonian, respectively:

$$H = \dot{x}^2/2 + (\delta+\varepsilon\cos t)x^2/2 \tag{1A}$$

$$H = \dot{x}^2/2 + (\delta+\varepsilon\cos t)x^2/2 + \varepsilon\alpha x^4/4 \tag{2A}$$

$$H = \dot{x}^2/2 + (\delta+\varepsilon y)x^2/2 + \varepsilon\alpha x^4/4 + \dot{y}^2/2 + y^2/2 \tag{3A}$$

We will use a perturbation procedure called von Zeipel's method [4], which is equivalent to terms of lowest order to the method of Lie transforms [5]. We will outline the procedure here but refer the reader to [6] for details.

For each of the systems (1A) - (3A) we will perform three canonical transformations, thereby obtaining an approximate first integral. We begin by transforming to so called action-angle I, θ variables for the $\varepsilon = 0$ problem. (This makes θ an ignorable coordinate for $\varepsilon = 0$.) Action-angle variables for the simple harmonic oscillator with frequency ω are given by

$$\dot{x} = \sqrt{2I\omega}\cos\theta \quad , \quad x = \sqrt{2I/\omega}\sin\theta \ .$$

The transformed Hamiltonians become respectively

$$(\dot{x},x) \to (I,\theta) \quad , \quad H = \sqrt{\delta}I + (\varepsilon/\sqrt{\delta})I\sin^2\theta\cos t \tag{1B}$$

$$(\dot{x},x) \to (I,\theta) \quad , \quad H = \sqrt{\delta}I + (\varepsilon/\sqrt{\delta})I\sin^2\theta\cos t + (\varepsilon\alpha/\delta)I^2\sin^4\theta \tag{2B}$$

$$\left.\begin{array}{l}(\dot{x},x)\to(I_1,\theta_1)\\ (\dot{y},y)\to(I_2,\theta_2)\end{array}\right\} , \ H = \sqrt{\delta}I_1+I_2+(\varepsilon/\sqrt{\delta})I_1\sqrt{2I_2}\sin^2\theta_1\sin\theta_2+(\varepsilon\alpha/\delta)I_1^2\sin^4\theta_1 \tag{3B}$$

Next we perform a near identity transformation to $\tilde{I}$, $\tilde{\theta}$ variables. This transformation is based upon the generating function [4]

$$F(\theta_i,\tilde{I}_i,t) = \sum_i \theta_i\tilde{I}_i + W(\theta_i,\tilde{I}_i,t)$$

where $W = O(\varepsilon)$ is yet to be determined. Then the transformation is given by

$$\tilde{\theta}_i = \frac{\partial F}{\partial \tilde{I}_i} = \theta_i + \frac{\partial W}{\partial \tilde{I}_i} \quad , \quad I_i = \frac{\partial F}{\partial \theta_i} = \tilde{I}_i + \frac{\partial W}{\partial \theta_i} \quad , \quad \tilde{H} = H + \frac{\partial F}{\partial t} = H + \frac{\partial W}{\partial t} \ .$$

Since $\tilde{\theta}_i = \theta_i + O(\varepsilon)$ we may replace θ_i by $\tilde{\theta}_i$ in W, neglecting terms of $O(\varepsilon^2)$. Then $\tilde{H} = \tilde{H}(\tilde{I}_i,\tilde{\theta}_i,t)$ and we select W so as to eliminate as many terms in $\tilde{H}$ which contain $\tilde{\theta}_i$ as possible. If δ is not close to $1/4$ all such terms may be eliminated with the result that the angles $\tilde{\theta}_i$ are ignorable and the actions $\tilde{I}_i$ are constants of the motion. In this case the system executes simple harmonic motion to $O(\varepsilon^2)$. However, when $\delta = 1/4$ a resonant term cannot be removed due to vanishing denominators.

To study the behavior of the system in the neighborhood of resonance we set

$$\sqrt{\delta} = 1/2 + \varepsilon\Delta \quad .$$

In this case the Hamiltonians become

$$\tilde{H} = \tilde{I}/2 + \varepsilon\{\Delta\tilde{I}-(\tilde{I}/2)\cos(2\tilde{\theta}-t)\} + O(\varepsilon^2) \qquad (1C)$$

$$\tilde{H} = \tilde{I}/2 + \varepsilon\{\Delta\tilde{I}-(\tilde{I}/2)\cos(2\tilde{\theta}-t)+3\alpha\tilde{I}^2/2\} + O(\varepsilon^2) \qquad (2C)$$

$$\tilde{H} = \tilde{I}_1/2 + \tilde{I}_2 + \varepsilon\{\Delta\tilde{I}_1+(\tilde{I}_1/2)\sqrt{2\tilde{I}_2}\sin(2\tilde{\theta}_1-\tilde{\theta}_2)+3\alpha\tilde{I}_1^2/2\} + O(\varepsilon^2) \qquad (3C)$$

Note that in (1C) and (2C) $\tilde{\theta}$ appears only as $2\tilde{\theta} - t$, while in (3C) $\tilde{\theta}_1$, $\tilde{\theta}_2$ appear only as $2\tilde{\theta}_1 - \tilde{\theta}_2$. This motivates our final canonical transformation which involves the new angle variables $\psi = \tilde{\theta} - t/2$, $\psi = \tilde{\theta}_1 - \tilde{\theta}_2/2$, respectively. After [7] we choose the following <u>linear</u> transformations which produce the corresponding Hamiltonians K:

$$\left.\begin{array}{l}\psi = \tilde{\theta} - t/2 \\ J = \tilde{I}\end{array}\right\} \begin{array}{ll} K = \varepsilon(\Delta J-(J/2)\cos 2\psi) + O(\varepsilon^2) & (1D) \\ K = \varepsilon(\Delta J-(J/2)\cos 2\psi+3\alpha J^2/2) + O(\varepsilon^2) & (2D)\end{array}$$

$$\left.\begin{array}{l}\psi_1=\tilde{\theta}_1-\tilde{\theta}_2/2,\ J_1=\tilde{I}_1 \\ \psi_2=\tilde{\theta}_2,\ J_2=\tilde{I}_2+\tilde{I}_1/2\end{array}\right\} K=J_2+\varepsilon(\Delta J_1+(J_1/2)\sqrt{2J_2-J_1}\sin 2\psi_1+3\alpha J_1^2/2)+O(\varepsilon^2) \qquad (3D)$$

In (1D), (2D) the Hamiltonian K is independent of time t and hence is a constant of the motion. In (3D) ψ_2 is ignorable and hence J_2 is constant. In this case K is also a constant and we take $K - J_2$ to be our constant of the motion for (3D). Transforming these first integrals back to the original coordinates, we obtain to $O(\varepsilon^2)$:

$$C_1 = 2\Delta(\dot{x}^2+x^2/4) - \{x\dot{x}\sin t+(\dot{x}^2-x^2/4)\cos t\} \qquad (1E)$$

$$C_2 = 2\Delta(\dot{x}^2+x^2/4) - \{x\dot{x}\sin t+(\dot{x}^2-x^2/4)\cos t\} + 3\alpha(\dot{x}^2+x^2/4)^2 \qquad (2E)$$

$$C_3 = 2\Delta(\dot{x}^2+x^2/4) - \{-x\dot{x}\dot{y}+(\dot{x}^2-x^2/4)y\} + 3\alpha(\dot{x}^2+x^2/4)^2 \qquad (3E)$$

In order to visualize the invariant surfaces (tori) which these equations represent, we consider their intersection with an appropriate surface of section Σ and display the corresponding invariant curves in the associated Poincaré map. This map represents successive intersections of an orbit with Σ [8].

In the case of systems (1), (2) which are set in x, $\dot{x}$, t space $(R^2\times S^1)$, we choose Σ: $t = 3\pi/2 (\text{mod } 2\pi)$. This produces the invariant curves

$$C_1 = 2\Delta(\dot{x}^2+x^2/4) + x\dot{x} \qquad (1F)$$

$$C_2 = 2\Delta(\dot{x}^2+x^2/4) + x\dot{x} + 3\alpha(\dot{x}^2+x^2/4)^2 \qquad (2F)$$

The shape of these curves depends Δ, α. See Fig. 1.

In the case of system (3) which is set in x, y, $\dot{x}$, $\dot{y}$ space (R^4), we choose Σ: $y = 0$. This produces the invariant curves

$$C_3 = 2\Delta(\dot{x}^2+x^2/4) + x\dot{x}\dot{y} + 3\alpha(\dot{x}^2+x^2/4)^2 \qquad (3F)$$

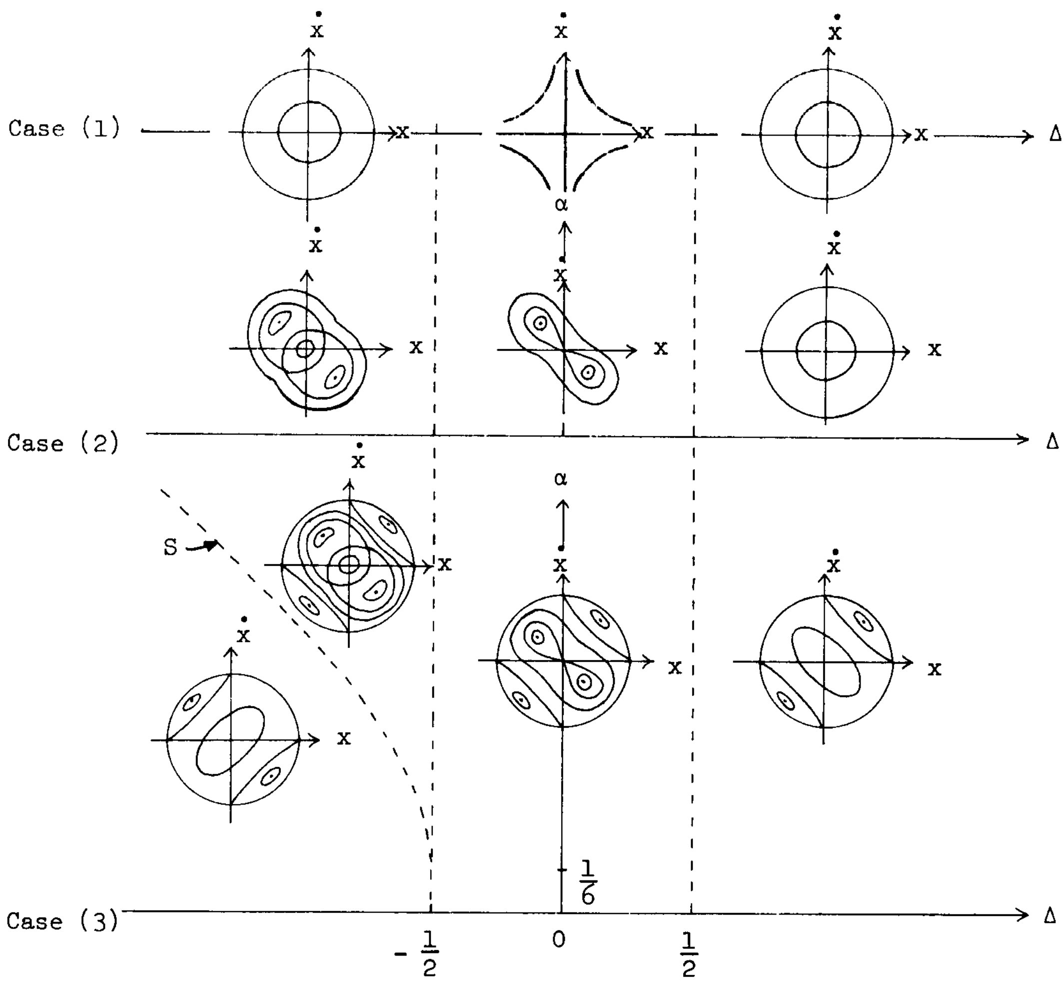

FIG. 1

Invariant curves in the Poincaré maps for systems (1) - (3) sketched for various Δ, α. A singular point in the Poincaré map corresponds to an equilibrium point or a periodic motion of the dynamical system. Note that the origin $x = \dot{x} = 0$ corresponds to the zero solution in all cases. Note also that system (1) corresponds to the Δ axis (α=0) of system (2). System (3) is displayed for $h = 1/2$.

These curves may be projected onto the $x\dot{x}$ plane by using the energy integral $H = h$, eq. (3A). To $O(\varepsilon)$ this gives

$$\dot{y} = \sqrt{2h-\dot{x}^2-x^2/4} \tag{3F'}$$

Eqs. (3F), (3F') are displayed in Fig. 1 for various Δ, α for $h = 1/2$. Note that for $\Delta < -1/2$ the qualitative nature of this family of curves changes as one crosses the bifurcation set S given in parametric form as [6]:

$$\Delta = \frac{-4+6v-3v^2}{8(1-v)^{3/2}} \quad , \quad \alpha = \frac{4-3v}{24(1-v)^{3/2}}$$

where $0 \leq v < 1$.

Discussion

Referring to Fig. 1, let us first compare systems (1) and (2). System (1) is a linearization of (2) in the neighborhood of the zero solution and hence the Poincaré map of (1) is a linearization of that of (2) in the neighborhood of $x = \dot{x} = 0$. Although the qualitative nature of the Poincaré maps for (1), (2) agree for $\Delta > 1/2$, they do not agree for $\Delta < 1/2$.

For $|\Delta| < 1/2$ both (1), (2) give the zero solution as unstable, but (1) predicts unbounded motions while (2) reveals the saddle of (1) to be part of a bounded separatrix. In particular (2) exhibits two centers which are both part of the same stable period 4π motion.

For $\Delta < -1/2$ the dynamical structure of (1) returns to the simple $\Delta > 1/2$ case while the Poincaré map for (2) exhibits two additional singular points which correspond to an unstable period 4π motion.

Let us now consider system (3). Note that while for systems (1) and (2) the domain of the Poincaré map is the entire $x\dot{x}$ plane, the motion of system (3) lies on the compact energy manifold (3A) which limits the domain of the Poincaré map to a closed and bounded region of the $x\dot{x}$ plane. Note also the appearance in the Poincaré map for (3) of two additional singular points near the bounding curve which corresponds to the "response mode" in which little of the total energy is in the motor. As we are interested in comparing (3) with (1), (2) we shall be mainly interested in the "motor mode" for which $x(t) \equiv 0$, and in the other periodic motions which bifurcate from it.

Consideration of Fig. 1 shows that a significant difference between systems (2), (3) lies in the bifurcation phenomenon associated with the set S of system (3) which is entirely absent from system (2). This may be explained by comparing eq. (2F) with eqs. (3F), (3F'). The extra factor $\dot{y}$ in the middle term of (3F) appears as unity in (2F). Indeed if $y = \cos t$ then $\dot{y} = -\sin t = 1$ on Σ: $t = 3\pi/2 \pmod{2\pi}$. Thus the expression (3F') for $\dot{y}$ is responsible for the differences between the Poincaré maps of (2), (3). This expression represents the transfer of energy from the response variable x to the motor variable y due to the coupling in system (3). Note that from (3A), α is the ratio of the nonlinearity coefficient $\varepsilon\alpha$ to the coupling coefficient ε. Thus from Fig. 1 when the coupling is small enough (for fixed nonlinearity and Δ), i.e. when α is large enough, system (3) behaves like system (2) in the region $\Delta < -1/2$. It is only when the coupling (i.e. the feedback from the "response" oscillator to the motor) is large that system (3) behaves differently from system (2).

Finally we note that these results are only approximate, not only because the perturbation method is valid only for small ε, but also because of the phenomenon of chaotic behavior (cf. KAM theory [9]). This latter effect involves a decrease in the measure of the set of motions which lie on invariant tori as ε is increased.

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