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Dynamics of Two Time-Delay Coupled Relaxation Oscillators of the van der Pol Type

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ABSTRACT

In this paper we examine the dynamics of two time-delay coupled relaxation oscillators of the van der Pol type. By integrating the governing differential-delay equations numerically, we find the various phase-locked motions including the in-phase and out-of-phase modes. Our computations reveal that depending on the strength of coupling (α) and the amount of time delay (τ), the in-phase (out-of-phase) mode may be stable or unstable. There are also values of α and τ for which the in-phase and out-of-phase modes are both stable leading to birhythmicity. The results are illustrated in the α - τ parameter plane. Near the boundaries of stability and instability of the in-phase (out-of-phase) mode, many other types of phase-locked motions can occur. Several examples of these phase-locked states are presented.

Introduction

A relaxation oscillator is characterized by a periodic motion which consists of a fast regime and a slow regime within each period. Coupled relaxation oscillators are often used to model the dynamics of a variety of physical, chemical and biological systems. There is a vast literature on this subject. In many of the applications the coupling process necessarily involves a time delay. Delays occur due to finite rates of transport such as diffusion and signal transmission. Despite this fact, effects of time delay have not been considered in the earlier works on coupled relaxation oscillators. Currently there is a great deal of interest in the study of relaxation oscillators with time-delay coupling. These studies deal with systems of neuronal oscillators that are coupled by a time delay through their synapses (Lee *et al.* [1997], Campbell and Wang [1998], and Yoshinga *et al.* [1999]).

In this paper we investigate the dynamics of a pair of relaxation oscillators of the van der Pol type with time-delay coupling. The motion of this system is governed by the following differential-delay equations:

$$\varepsilon \ddot{x}_1 - (1 - x_1^2) \dot{x}_1 + x_1 = \alpha [x_2(t - \tau) - x_1(t - \tau)] \quad (1a)$$

$$\varepsilon \ddot{x}_2 - (1 - x_2^2) \dot{x}_2 + x_2 = \alpha [x_1(t - \tau) - x_2(t - \tau)] \quad (1b)$$

Here x_1 and x_2 are the dependent variables (e.g., voltages), t is time, and a dot over a variable denotes a time-derivative. The coupling strength and

time delay are designated by the symbols α and τ , respectively, and the parameter ε is assumed to be much less than unity.

In an earlier work, Storti and Rand [1986] examined the above equations in the absence of time delay (i.e., $\tau = 0$). Using the method of matched asymptotic expansions and Floquet theory, they determined the conditions for stability of the in-phase and out-of-phase modes. They showed that in the limit $\varepsilon \rightarrow 0$, the in-phase mode is stable for $-1/2 < \alpha < 0$ and $\alpha > 1$, and is unstable for $\alpha < -1/2$ and $0 < \alpha < 1$. The out-of-phase mode, on the other hand, is found to be stable when $-1/2 < \alpha < -1/3$ and $\alpha > 0$; it is unstable for $-1/3 < \alpha < 0$, whereas for $\alpha < -1/2$, the out-of-phase mode does not exist. This work was extended more recently by Storti and Reinhall [1997] to include the effect of velocity coupling.

Grasman [1987] considered the time-delayed system (1) and reformulated the equations in terms of the Lienard variables as follows.

$$\varepsilon \dot{x}_1 = y_1 + x_1 - \frac{x_1^3}{3} \quad (2a)$$

$$\dot{y}_1 = -x_1 + \alpha [x_2(t - \tau) - x_1(t - \tau)] \quad (2b)$$

$$\varepsilon \dot{x}_2 = y_2 + x_2 - \frac{x_2^3}{3} \quad (2c)$$

$$\dot{y}_2 = -x_2 + \alpha [x_1(t - \tau) - x_2(t - \tau)] \quad (2d)$$

He analyzed these equations for weak coupling by

assuming that $0 < \varepsilon \ll \alpha < 1$. Under this condition the coupling does not perturb the limit cycle of the individual oscillators; it only alters their phases. Grasman [1987] derived solutions for the phase functions in order to describe the dynamics of the coupled system. In this study we examine Eqns (1) and (2) for a range of values of the coupling strength and time delay. The objective is to determine the possible phase-locked motions that this system can exhibit and find the values of α and τ for which such motions can occur. Our approach is based on a numerical solution of these equations. The computational procedure we have used is discussed in detail by Hairer *et al.* [1987].

The results of this study may be summarized as follows. The time-delayed system given by Eqn (1) or Eqn (2) undergoes a variety of phase-locked motions including an in-phase mode and an out-of-phase mode. Depending on the coupling strength and the amount of time delay, the in-phase (out-of-phase) mode may be stable or unstable. We also find that for certain values of these parameters the in-phase and out-of-phase modes are both stable leading to birhythmicity. The results are illustrated in the $\alpha - \tau$ parameter plane. Near the boundaries of stability and instability of the in-phase (out-of-phase) mode, many other types of phase-locked states may exist. Several examples of these phase-locked states are presented.

The In-Phase and Out-of-Phase Modes: Existence and Stability

It is easy to see that for all values of α and τ , Eqns (1) and (2) admit the solution: $x_1 = x_2 = u(t)$, representing the in-phase mode. Here $u(t)$ describes the relaxation limit cycle of the uncoupled van der Pol oscillator which is governed by the equation

$$\varepsilon \ddot{u} - (1 - u^2)\dot{u} + u = 0. \quad (3)$$

Equation (3) has been studied in great detail by many researchers. Here we simply note that for $\varepsilon = 0.05$ which is the value used in all our computations, this period is approximately equal to 2.43 (see, for example, Grasman [1987]).

Equations (1) and (2) also admit the out-of-phase solution: $x_1 = -x_2 = v(t)$, where $v(t)$ satisfies the following equation.

$$\varepsilon \ddot{v} - (1 - v^2)\dot{v} + v = -2\alpha v(t - \tau). \quad (4)$$

For given values of α and τ , this equation deter-

mines the existence of the out-of-phase mode. In order to investigate the stability of the in-phase and out-of-phase modes we integrate Eqn (2) numerically. Consider first the in-phase mode. To find the values of α and τ for which the in-phase mode is stable, we proceed as follows. Choose a value of $\varepsilon \ll 1$; as mentioned above we have used $\varepsilon = 0.05$. Now choose the parameters α and τ , and select an initial condition which is close to the in-phase mode: $x_1 = x_2$. We have used $x_1 = 1.3$, $x_2 = 1.35$, with $y_1 = y_2 = 0$. Next we integrate the uncoupled equations in (2), i.e., with $\alpha = 0$, backward from $t = 0$ to $t = -\tau$. The solution thus obtained is subsequently used as the set of initial conditions over the interval $[-\tau, 0]$ to integrate the coupled time-delay equations in (2). During each step of this integration we must take into account the values of (x_1, x_2, y_1, y_2) at a time τ units earlier. We have adopted a fourth order Runge-Kutta method for our integration. The details of this technique are given in Hairer *et al.* [1987]. The stability of the in-phase mode is established by observing if the computed trajectories in the four dimensional space approach the in-phase mode with time.

To investigate the stability of the out-of-phase mode, we choose the parameters α and τ , and use an initial condition close to this mode. In our simulation we have used $x_1 = 1.3$, $x_2 = -1.35$, and $y_1 = y_2 = 0$. Next we integrate the uncoupled equations in (2) from $t = 0$ to $t = -\tau$, analogous to the situation with the in-phase mode. Using this solution as the set of initial conditions over the interval $[-\tau, 0]$, a numerical solution of the differential-delay equations (2) is found. The above calculations are repeated for several values of α and τ . The results are depicted in Figures 1 and 8 for the in-phase and out-of-phase modes, respectively.

Results and Discussion

Figure 1 reveals that for a given value of α with $\alpha > 0.68$, the in-phase mode remains stable for small delays, but it loses stability when the delay exceeds a critical value given by the transition curve. For α lying in the interval $-0.5 < \alpha < 0$, the in-phase mode remains stable for much longer delays beyond which it becomes unstable. There are also several thin regions around the τ -axis (i.e.,

for small α) within which the in-phase mode is stable. These regions alternate on the positive and negative sides of the τ -axis with each region spanning approximately half the period of the uncoupled oscillators. The in-phase mode is unstable in the remainder of the α - τ plane. The inset in Figure 1 shows the transition curve over the interval $\alpha > 0.68$ in more detail. Near the boundaries of stability and instability of the in-phase mode, many other types of phase-locked states may exist. Consider, for instance, the values of α and τ in the vicinity of the point A on the transition curve (see the inset in Figure 1). In particular, consider $\tau = 0.16$ and several values of α , namely, $\alpha = 1.15, 1.2, 1.205$ and 1.208 . Figure 2 depicts the phase-locked modes that exist at these parameter values. Note that since Eqns (1) are invariant under the transformation which interchanges x_1 and x_2 , each of the periodic motions which is displayed in Figures 2-7 and Figures 9-11, is accompanied by a periodic motion which is its reflection in the line $x_1 = x_2$. Next consider $\alpha = 1.208$ and $\tau = 0.16, 0.162, 0.165$ and 0.17 . The various phase-locked states for these parameters are illustrated in Figure 3. We now fix $\tau = 0.16$ and see if there are other values of α at which phase-locked modes different from the in-phase and out-of-phase modes can exist. We find that for α lying between 2.5 and 3.3, many such phase-locked motions can occur. This is the interval with $\tau = 0.16$ over which the transition curve has an upward concavity. Figure 4 portrays the phase-locked mode corresponding to $\alpha = 3.0$ and $\tau = 0.16$. Consider another interval in the α - τ plane in which the transition curve is concave upward. In particular, consider the horizontal line given by $\tau = 0.115$ with values of α between 4.8 and 5.7. For every point on this line there exists a phase-locked state which is different from the in-phase and the out-of-phase modes. The phase-locked state corresponding to $\alpha = 5.0$ and $\tau = 0.115$ is displayed in Figure 5. Another interval with upward concavity lies over $\tau = 0.085$ and $7.1 < \alpha < 7.9$. Figure 6 shows the phase-locked motion at the point given by $\alpha = 7.7$ and $\tau = 0.085$ lying in this interval. As a final example, consider the point with

parameter values $\alpha = -0.5$ and $\tau = 0.75$, which lies near the boundary between stability and instability on the negative side of the α -axis. The phase-locked mode for this case is shown in Figure 7. A variety of other phase-locked motions can occur near the boundaries of stability and instability of the in-phase mode. For the sake of brevity we will not discuss them further.

The numerical results for the stability of the out-of-phase mode are displayed in Figure 8. This figure reveals that for positive coupling strength ($\alpha > 0$), the out-of-phase mode is stable for all values of τ except in thin regions adjacent to the τ -axis where it is unstable. When the coupling strength is negative ($\alpha < 0$), there are also thin regions adjacent to the τ -axis where the out-of-phase mode is unstable. Lying to the left of these thin regions and to the right of the line $\alpha = -0.5$, there is a stable region for the out-of-phase mode. At $\alpha = -0.5$, the out-of-phase mode undergoes an infinite period bifurcation in which $x_1(t) = -x_2(t)$ remains constant with time. This constant solution can also be readily observed from Eqn (4) with $\alpha = -0.5$. For α less than this value, the out-of-phase mode ceases to exist. Analogous to the situation in Figure 1, other phase-locked modes are found near the boundaries of stability and instability of the out-of-phase mode. Figure 9 shows the phase-locked mode corresponding to $\alpha = 0.05$ and $\tau = 1.25$. Another phase-locked mode which occurs for $\alpha = -0.025$ and $\tau = 2.7$ is shown in Figure 10. Near the boundary line $\alpha = -0.5$, other types of phase-locked motions are found. Figure 11 depicts such a phase-locked mode for $\alpha = -0.51$ and $\tau = 1.5$. As seen from this figure, this phase-locked mode consists of oscillatory motions of large and small amplitudes with a nonzero mean.

By superimposing Figures 1 and 8, we see that there are several overlapping regions in the α - τ parameter plane where the in-phase and out-of-phase modes are both stable. These regions of birhythmicity are illustrated in Figure 12. Around the τ -axis, Yeung and Strogatz [1999] found similar overlapping regions between the synchronized and incoherent states in their study of the Kuramoto model of oscillators with time-delay coupling. Wirkus and Rand [1999] found parameter

regions in which both the in-phase and out-of-phase modes are stable in a study of two van der Pol oscillators coupled with time delay (in the limit of non-relaxation behavior).

Concluding Remarks

We have examined the dynamics of two identical relaxation oscillators of the van der Pol type which are coupled by time delay. By integrating the governing differential-delay equations numerically, we have found the various phase-locked motions. The results are presented in the $\alpha - \tau$ parameter plane. Our computations reveal that depending on the strength of coupling (α) and the amount of time delay (τ), a stable in-phase mode or a stable out-of-phase mode may exist. For other values of these parameters the in-phase or the out-of-phase mode may become unstable. We also found that near the boundaries of stability and instability of these modes in the $\alpha - \tau$ parameter plane, many other phase-locked motions can occur. Several examples of these phase-locked states have been presented.

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Figure Captions

Figure 1. Stability results for the in-phase mode in the $\alpha - \tau$ parameter plane for $\varepsilon = 0.05$. S = Stable, U = Unstable.

Figure 2. Phase-locked states near the point A on the inset of Figure 1. These states correspond to $\tau = 0.16$ and (a) $\alpha = 1.15$, (b) $\alpha = 1.2$, (c) $\alpha = 1.205$ and (d) $\alpha = 1.208$, respectively.

Figure 3. Phase-locked states near the point A on the inset of Figure 1. These states correspond to $\alpha = 1.208$ and (a) $\tau = 0.16$, (b) $\tau = 0.162$, (c) $\tau = 0.165$ and (d) $\tau = 0.17$, respectively.

Figure 4. Phase-locked state for $\alpha = 3$ and $\tau = 0.16$ in Figure 1.

Figure 5. Phase-locked state for $\alpha = 5$ and $\tau = 0.115$ in Figure 1.

Figure 6. Phase-locked state for $\alpha = 7.7$ and $\tau = 0.085$ in Figure 1.

Figure 7. Phase-locked state for $\alpha = -0.5$ and $\tau = 0.75$ in Figure 1.

Figure 8. Stability results for the out-of-phase mode in the $\alpha - \tau$ parameter plane for $\varepsilon = 0.05$. S = Stable, U = Unstable, N = Does not exist.

Figure 9. Phase-locked state for $\alpha = 0.05$ and $\tau = 1.25$ in Figure 8.

Figure 10. Phase-locked state for $\alpha = -0.35$ and $\tau = 0.22$ in Figure 8.

Figure 11. Phase-locked state for $\alpha = -0.51$ and $\tau = 1.7$ in Figure 8. (a) x_1-x_2 plane, (b) time series for $x_1(t)$ and $x_2(t)$.

Figure 12. Regions of birhythmicity in the $\alpha-\tau$ parameter plane for $\varepsilon = 0.05$. These regions are shown shaded in the figure.

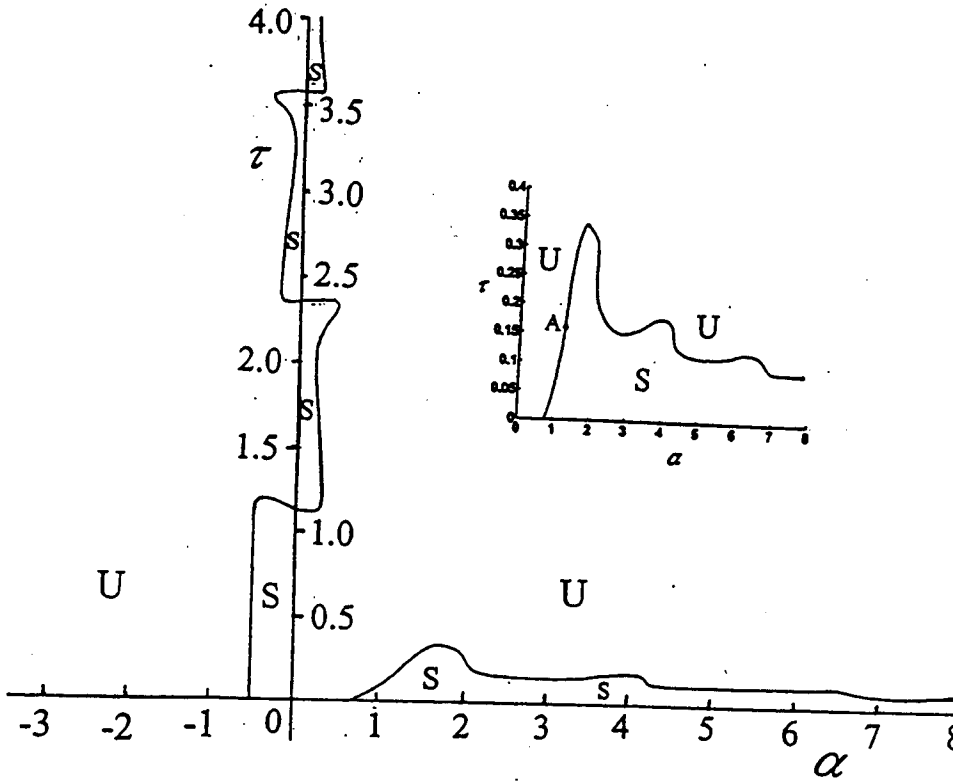


Figure 1

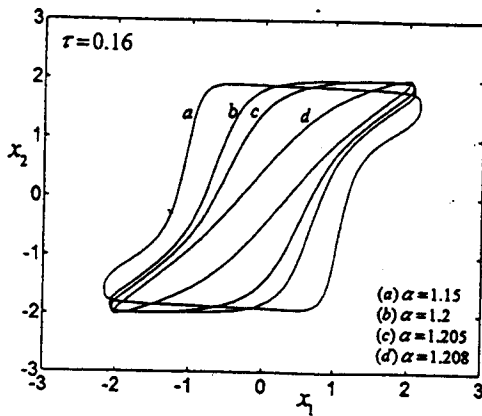


Figure 2

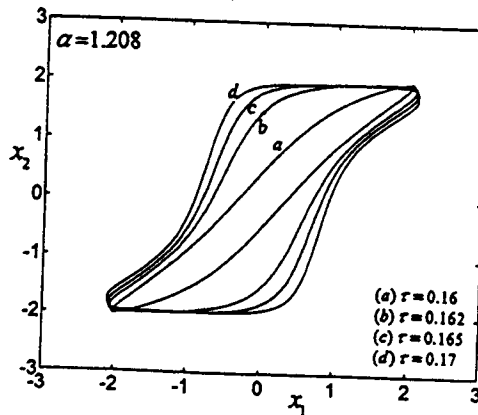


Figure 3

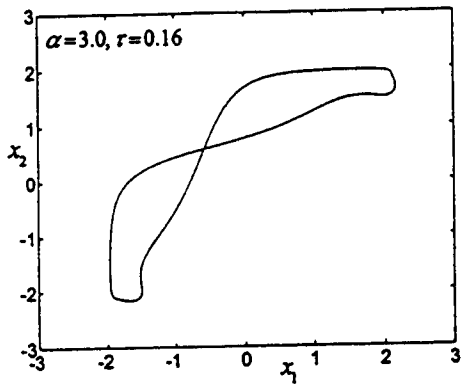


Figure 4

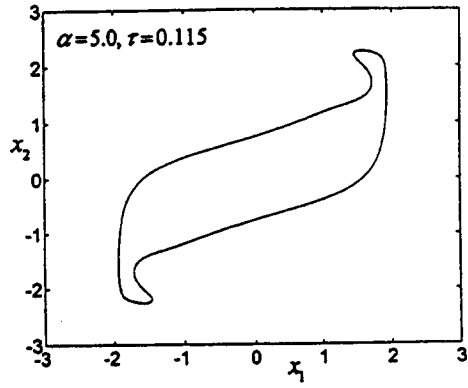


Figure 5

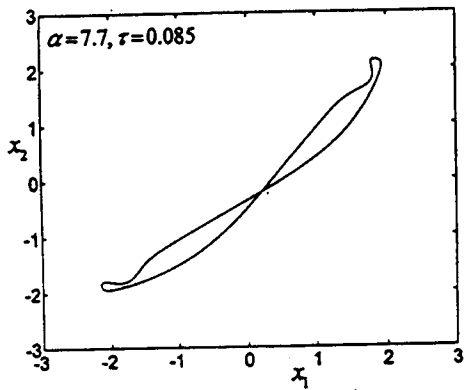


Figure 6

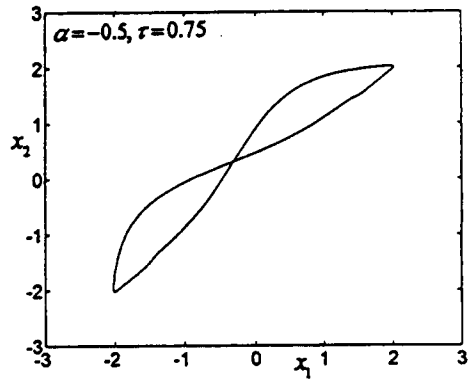


Figure 7

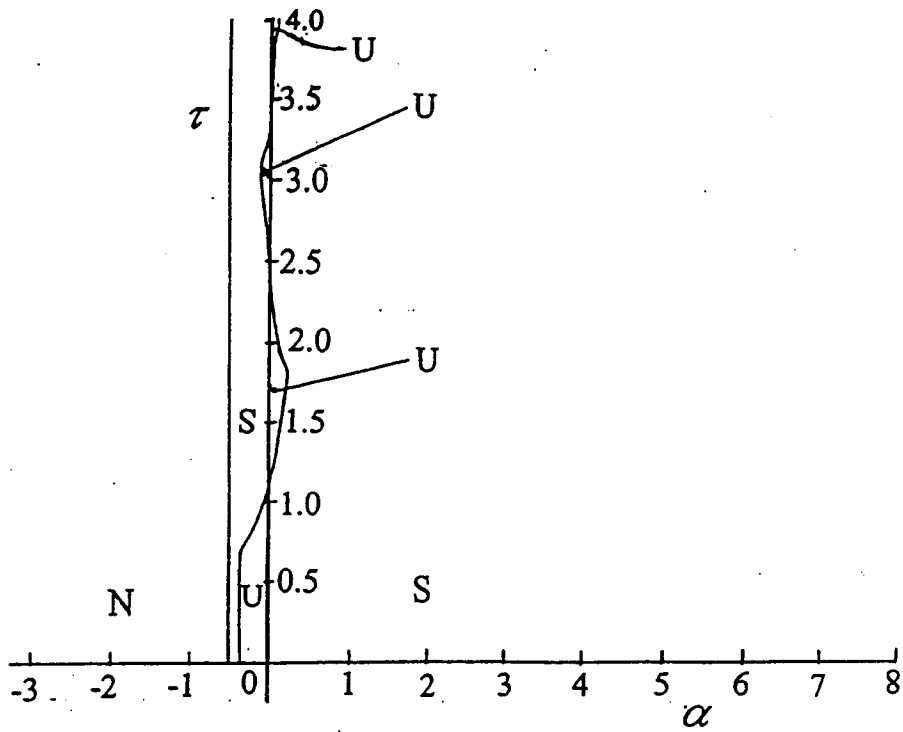


Figure 8

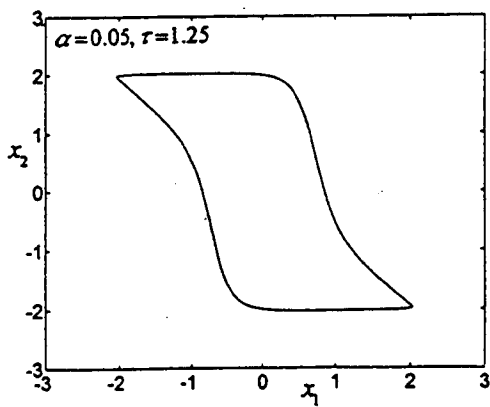


Figure 9

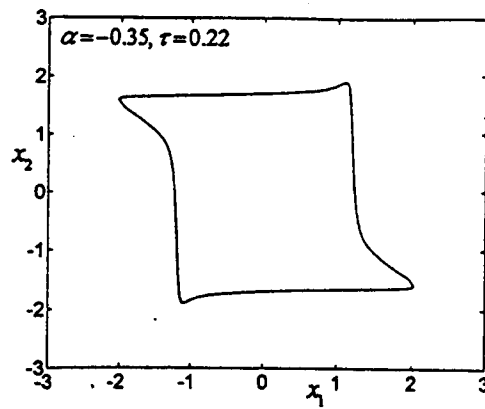


Figure 10

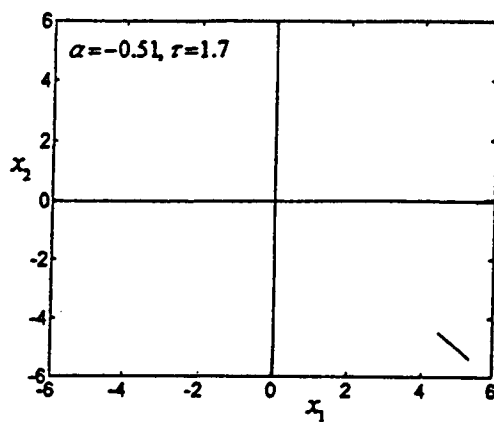


Figure 11(a)

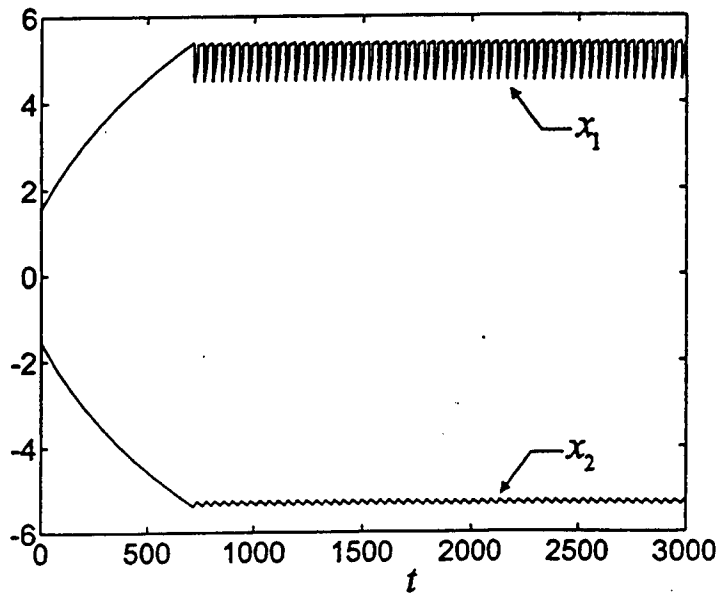


Figure 11(b)

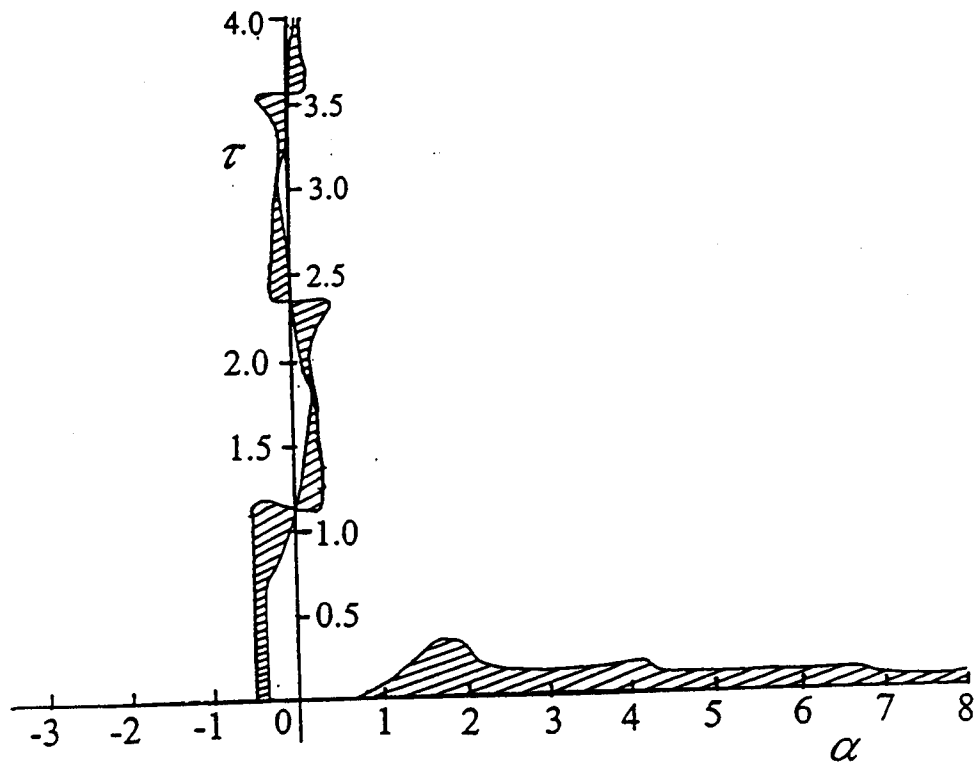


Figure 12