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Chapter 8
Coupled Parametrically Driven Modes in Synchrotron Dynamics

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Abstract This work concerns the dynamics of a type of particle accelerator called a synchrotron, in which particles are made to move in nearly circular orbits of large radius. The stability of the transverse motion of such a rotating particle may be modeled as being governed by Mathieu's equation. For a train of two such particles the equations of motion are coupled due to plasma interactions and resistive wall coupling effects.

In this paper we study a system consisting of a train of two such particles which is modeled as two coupled nonlinear Mathieu equations with delay coupling. In particular we investigate the stability of two coupled parametrically forced linear normal modes.

Keywords Parametric excitation • Coupled oscillators • Mathieu equation • Stability analysis • Synchrotron

8.1 Introduction

The synchrotron is a particle accelerator in which a "particle" actually consists of a group of electrons in a "bunch." We ignore the interactions of electrons inside each bunch and treat the entire bunch as a single particle.

Each bunch leaves an electrical disturbance behind it as it traverses around the synchrotron, and these wake fields are the main source of coupling in the model. The coupling is mediated by several sources, including resistive wall coupling, ion coupling, and the electron cloud effect. The wake fields can persist through entire orbits.

The ensemble of all bunches is called a "train." When modeled as several interacting bunches, each bunch is coupled to the wake of the bunch in front of it. When modeled as a train, the train interacts with itself after a full orbit. We will combine both of these into one cohesive model.

The particle is made to move in a circle-like orbit through the use of about 100 magnets. The magnets are an external force dictating the path of the electron, and the circular nature of this path means that the forcing is periodic in rotation angle θ; using $\theta = \omega t$, the forcing is periodic in time as well. We can express this forcing function as a Fourier series, and we shall approximate this series by the first couple of terms in it, namely the constant term and the first cosine term.

We model each bunch as a scalar variable $x_i(t), i = 1, \ldots, n$. Here x_i is the vertical displacement above equilibrium of the center of mass of the ith bunch. Each x_i is modeled as an oscillator, and we write:

$$\begin{aligned} \ddot{x}_1 + (\delta + \epsilon \cos t)\, x_1 + \epsilon\gamma x_1^3 &= \beta \sum_{j=1}^{n} x_j(t-T) \\ \ddot{x}_i + (\delta + \epsilon \cos t)\, x_i + \epsilon\gamma x_i^3 &= \beta \sum_{j=1}^{n} x_j(t-T) + \alpha x_{i-1}, \quad i = 2, \ldots, n \end{aligned} \tag{8.1}$$

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When modeling the train as a single unit, we are concerned with the forces on it due to the disturbance left at a given point by itself, one cycle ago. This effect will be modeled as a delay term, where the delay is the time to rotate through one cycle. β represents the strength of this coupling (Meller, personal communication to author RR on 12-27-13).

When modeling the interaction between bunches, each bunch influences the bunch behind it in the train. Since the time for a bunch to move to the position occupied by the previous bunch is small relative to the time needed to traverse the entire circumference of the synchrotron, the bunch-to-bunch interactions are not modeled with delay terms. α represents the strength of this coupling (Meller, personal communication to author RR on 12-27-13).

The simplest case is a single bunch, $n = 1$. Here we only consider the effect of delayed self feedback:

$$\ddot{x}_1 + (\delta + \epsilon\nu\cos t)x_1 = \beta x_1(t - T) \tag{8.2}$$

A system of this type has been investigated by Morrison and Rand [1]. It was shown that the region of instability associated with 2:1 subharmonic resonance can be eliminated by choosing the delay T long enough.

8.2 Model 1

In this paper we study a system of two bunches (Eq. (8.1) with $n = 2$), in which we ignore nonlinear terms as well as delay. The equations are:

$$\begin{aligned} \ddot{x} + (\delta + \epsilon\cos t)\, x &= \beta(x + y) \\ \ddot{y} + (\delta + \epsilon\cos t)\, y &= \beta(x + y) + \alpha x \end{aligned} \tag{8.3}$$

We uncouple these equations with the following linear transformation:

$$x = u + v, \qquad y = \sqrt{\frac{\beta + \alpha}{\beta}}(u - v) \tag{8.4}$$

giving

$$\begin{aligned} \ddot{u} + (\Omega_u^2 + \epsilon\cos t)\, u &= 0 \\ \ddot{v} + (\Omega_v^2 + \epsilon\cos t)\, v &= 0 \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} \Omega_u^2 &= \delta - \beta - \sqrt{\beta(\beta + \alpha)} \\ \Omega_v^2 &= \delta - \beta + \sqrt{\beta(\beta + \alpha)} \end{aligned} \tag{8.6}$$

The system has thus been reduced to a pair of uncoupled Mathieu equations, each of the form:

$$\ddot{z} + (\Delta + \epsilon\cos t)\, z = 0 \tag{8.7}$$

As is well known, this equation exhibits a 2:1 subharmonic resonance in the neighborhood of $\Delta = 1/4$. The boundaries of the associated tongue of instability are given by:

$$\Delta = \frac{1}{4} \pm \frac{\epsilon}{2} + O(\epsilon^2) \tag{8.8}$$

Comparison of Eqs. (8.6) and (8.8) gives the instability tongues as $\Omega_i^2 = \frac{1}{4} \pm \frac{\epsilon}{2}$, or in terms of the parameter δ, the two tongues become:

$$\delta = \frac{1}{4} + \beta \pm \sqrt{\beta(\beta + \alpha)} \pm \frac{\epsilon}{2} + O(\epsilon^2) \tag{8.9}$$

$$\delta_0 = \frac{1}{4} + \beta \pm \sqrt{\beta(\beta + \alpha)} \tag{8.10}$$

$$\delta_1 = \pm\frac{1}{2} \tag{8.11}$$

It is clear from this result that the introduction of α and β only affects δ_0. Graphically, this means that as the parameters α, β change, the tongue of instability is translated left or right in the δ-ϵ plane, and the slopes of the transition curves remain the same.

Four graphs of the transition curves are shown on the next page. The shaded regions are unstable and the unshaded regions are stable.

8.3 Model 2

Another model we investigated was one where the coupling terms were O(ϵ). In this model, the equations are:

$$\begin{aligned} \ddot{x} + (\delta + \epsilon\cos t)\, x &= \epsilon\beta(x + y) \\ \ddot{y} + (\delta + \epsilon\cos t)\, y &= \epsilon\beta(x + y) + \epsilon\alpha x \end{aligned} \tag{8.12}$$

We will employ the technique of harmonic balance to find expressions for the transition curves to order ϵ [2]. We assume a solution can be found of the form:

$$\begin{aligned} x &= A\cos\tfrac{t}{2} + B\sin\tfrac{t}{2} \\ y &= C\cos\tfrac{t}{2} + D\sin\tfrac{t}{2} \end{aligned} \tag{8.13}$$

And substitute these equations into (8.12).

After using some trig identities we obtain:

$$\begin{aligned} 0 = {} & \frac{\epsilon B}{2}\sin\frac{3t}{2} + \frac{\epsilon A}{2}\cos\frac{3t}{2} \\ & + \left(B\delta - \epsilon\beta D - \epsilon\beta B - \frac{\epsilon B}{2} - \frac{B}{4}\right)\sin\frac{t}{2} \\ & + \left(A\delta - \epsilon\beta C - \epsilon\beta A + \frac{\epsilon A}{2} - \frac{A}{4}\right)\cos\frac{t}{2} \end{aligned}$$

$$\begin{aligned} 0 = {} & \frac{\epsilon D}{2}\sin\frac{3t}{2} + \frac{\epsilon C}{2}\cos\frac{3t}{2} \\ & + \left(D\delta - \epsilon\beta D - \epsilon\beta B - \epsilon\alpha B - \frac{\epsilon D}{2} - \frac{D}{4}\right)\sin\frac{t}{2} \\ & + \left(C\delta - \epsilon\beta C - \epsilon\beta A - \epsilon\alpha A + \frac{\epsilon C}{2} - \frac{C}{4}\right)\cos\frac{t}{2} \end{aligned}$$

Since we only care about O(ϵ) we disregard the $\cos\frac{3t}{2}$ and $\sin\frac{3t}{2}$ terms. Taking the coefficients of the remaining trigonometric functions to be zero, we obtain four equations in A, B, C, and D. Writing this as a matrix, we get

$$\begin{bmatrix} -\frac{(4\beta-2)\epsilon-4\delta+1}{4} & 0 & -\epsilon\beta & 0 \\ -(\alpha+\beta)\epsilon & 0 & -\frac{(4\beta-2)\epsilon-4\delta+1}{4} & 0 \\ 0 & -\frac{(4\beta+2)\epsilon-4\delta+1}{4} & 0 & -\epsilon\beta \\ 0 & -(\alpha+\beta)\epsilon & 0 & -\frac{(4\beta+2)\epsilon-4\delta+1}{4} \end{bmatrix} \tag{8.14}$$

For this system to have a nontrivial solution, the determinant must be zero. This produces the equation:

$$-\frac{((4\beta-2)\epsilon-4\delta+1)^2\left(\frac{((4\beta+2)\epsilon-4\delta+1)^2}{16}-\beta(\alpha+\beta)\epsilon^2\right)}{16}$$
$$+\beta(\alpha+\beta)\epsilon^2\left(\frac{((4\beta+2)\epsilon-4\delta+1)^2}{16}-\beta(\alpha+\beta)\epsilon^2\right)=0 \tag{8.15}$$

With the solution:

$$\delta=\frac{1}{4}+\epsilon\left(\beta\pm\sqrt{\beta(\beta+\alpha)}\pm\frac{1}{2}\right)+O(\epsilon^2) \tag{8.16}$$

As in Model 1, we write δ in the form $\delta=\delta_0+\epsilon\delta_1+O(\epsilon^2)$ to obtain

$$\delta_0=\frac{1}{4} \tag{8.17}$$

$$\delta_1=\beta\pm\sqrt{\beta(\beta+\alpha)}\pm\frac{1}{2} \tag{8.18}$$

Unlike in Model 1, here the introduction of α and β only affects δ_1. Graphically, this means that as the parameters α, β change, the tongue of instability narrows or widens while intersecting the same point on the δ-axis.

Four graphs of the transition curves are shown on the next page. The shaded regions are unstable and the unshaded regions are stable.

8.4 Results

There are a few important observations to make about the tongues of instability.

8.4.1 Model 1

The first is that a small stable region appears between the tongues of instability, and it grows in size as both parameters α and β increase. Thus, while most of the graph becomes more unstable as the coupling increases, there is a region where the system actually becomes more stable.

The second observation to note is how these transition curves change with respect to both parameters. When β is increased, one of the tongues stays relatively still while the other tongue moves to the right. This causes the intersection of the tongues to move both up and to the right. When α is increased, the tongue on the left moves further to the left and the tongue on the right moves further to the right. This movement is balanced so that the intersection of the tongues moves straight up.

8.4.2 Model 2

Unlike in Model 1, there is no small stable region in Model 2 that grows in size with the parameters α, β. More explicitly, there is no region that becomes more stable as α and β are increased; all unstable regions stay unstable as the parameters increase.

8.5 Conclusion

In this paper we showed the distinct effects that the two types of coupling, α and β, have on the system. Furthermore, we showed that both effects are relevant to the problem and are worth considering when mathematically modeling the synchrotron, and that the nature of the instability depends on the way in which the coupling is modeled.

Note that both models give the same result when ϵ is fixed and α, β vary. As an example, suppose $\epsilon = 0.1$ and we take $\alpha = \beta = 0.01$ in Model 1. Then the values of δ where the stability changes are given by $\delta_{crit} = (0.1957, 0.2958)$. For Model 2 we'd have to use the parameter values $\alpha = 0.1$, $\beta = 0.1$, since the coupling is $\epsilon\alpha$, $\epsilon\beta$ and $\epsilon = 0.1$. Using these values in Model 2 gives the same δ_{crit} values as the ones in Model 1.

The difference between these models comes when the parameters α, β are fixed and ϵ is allowed to vary. In this case we get the transition curves shown in Figs. 8.1 and 8.2.

For future papers we will explore the effects of delay and nonlinearity on this model.

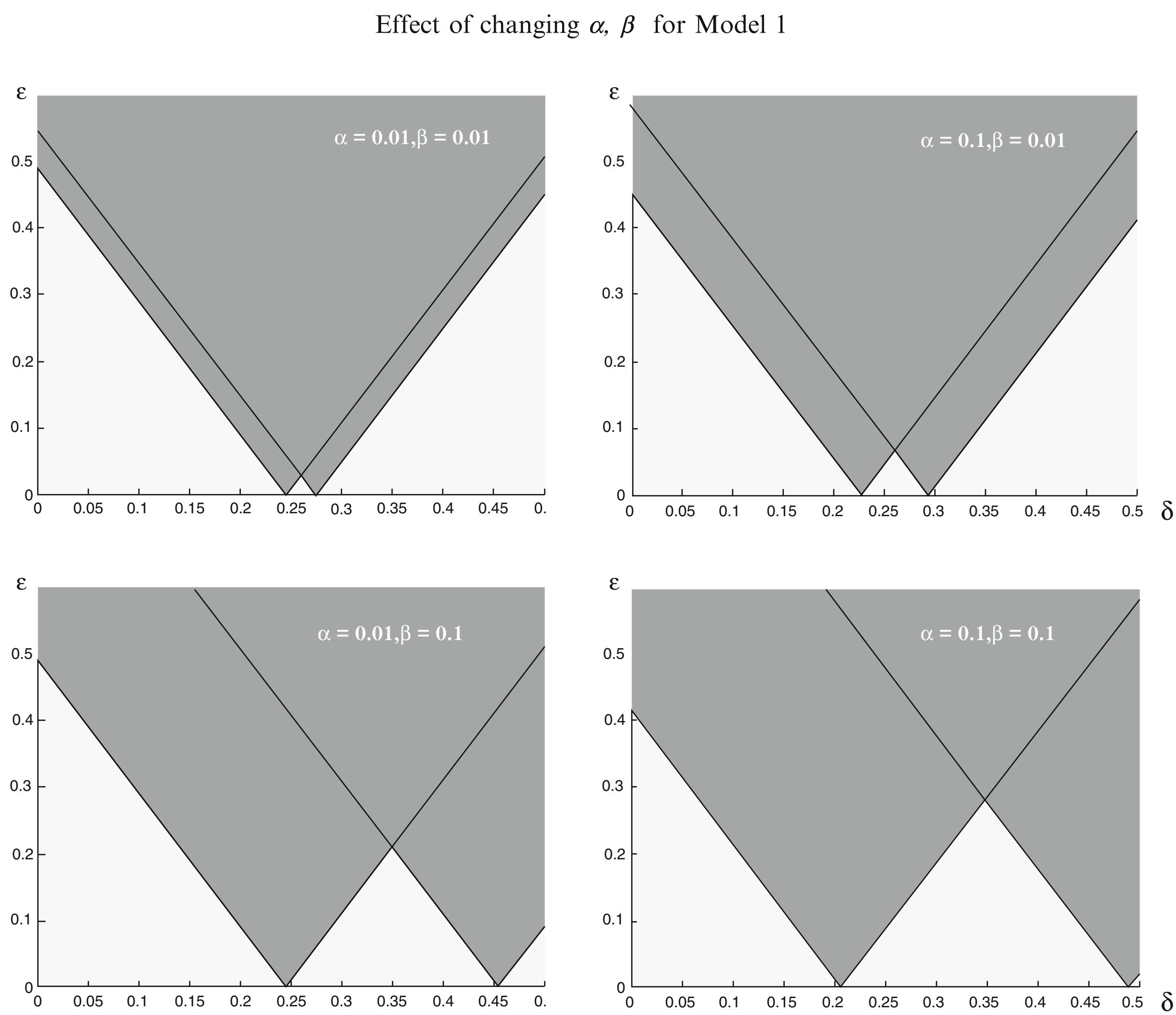

Fig. 8.1 *Shaded regions* are unstable, *unshaded regions* are stable

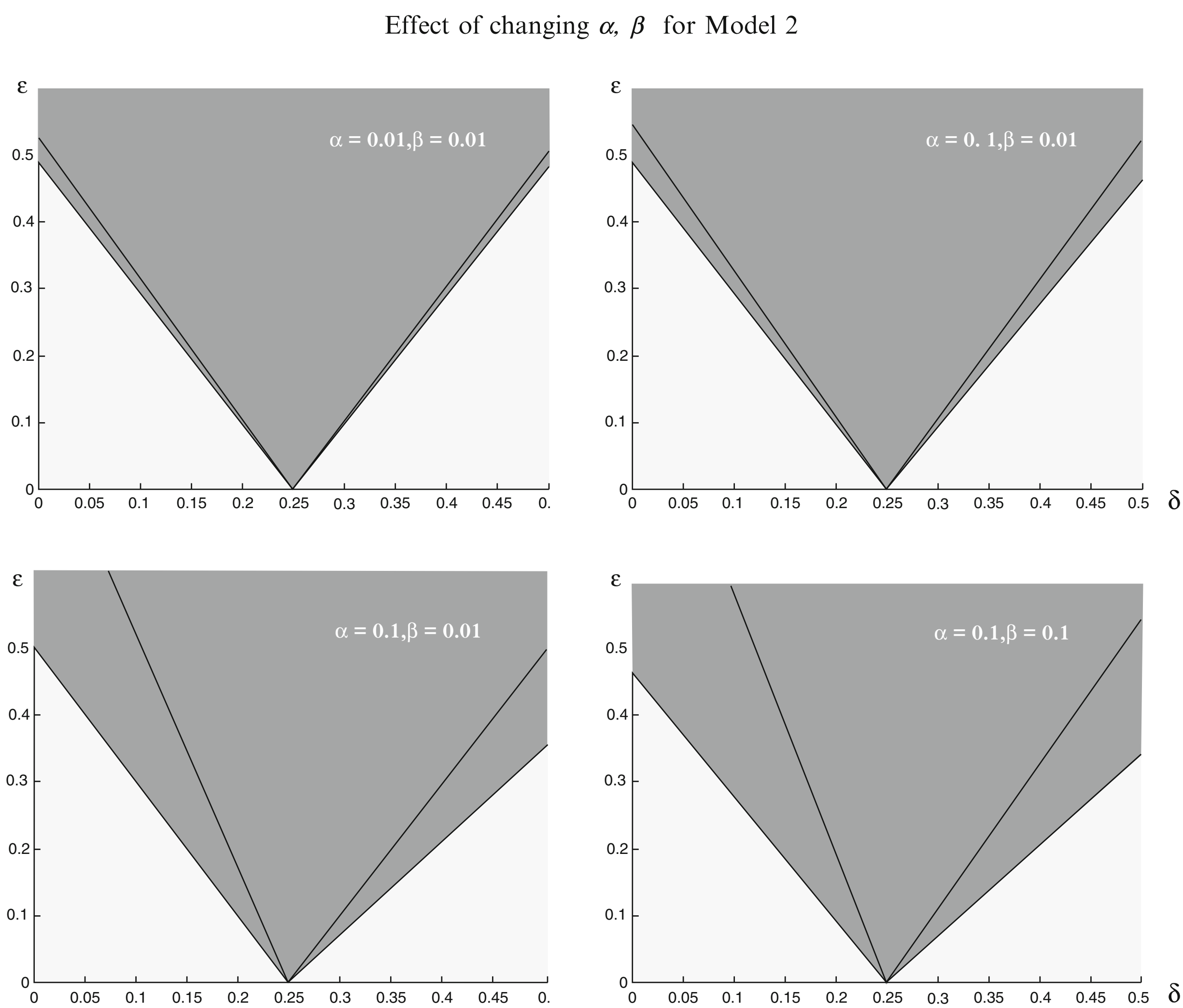

Fig. 8.2 *Shaded regions* are unstable, *unshaded regions* are stable

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