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Short communication

A digital model of coupled oscillators

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ABSTRACT

A new model of coupled oscillators is proposed and investigated. All phase variables and parameters are integer-valued. The model is shown to exhibit two types of motions, those which involve periodic phase differences, and those which involve drift. Traditional dynamical concepts such as stability, bifurcation and chaos are examined for this class of integer-valued systems. Numerical results are presented for systems of two and three oscillators. This work has application in digital technology.

1. Introduction

A common model for coupled oscillators takes the form [4,11,1–3,10,9,5,8,6]

$$\frac{d}{dt}\theta_i = \omega_i + \alpha\sum_{k=1}^{N}\sin(\theta_k - \theta_i), \quad i = 1, 2, \ldots, N \tag{1}$$

Here t is time, represented as a continuously flowing real number, $\theta_i(t)$ is a real number representing the phase of oscillator i at time t, ω_i is the uncoupled frequency of oscillator i, and α is a coupling constant. In this paper we consider the following extension of Eq. (1), in which all quantities are integers:

$$\theta_i^{n+1} = \theta_i^n + \omega_i + \alpha\sum_{k=1}^{N}\operatorname{sgn}(\theta_k^n - \theta_i^n), \quad i = 1, 2, \ldots, N \tag{2}$$

Here time n is treated as an integer $\{0, 1, 2, 3, \ldots\}, \theta_i^n$ is an integer representing the phase of oscillator i at time n, and quantities ω_i and α have the same meaning as they did in Eq. (1), except here they are integers.

Applications of this integer model occur in digital technology, including synchronization and digital communications [12,7]. For example, a system of coupled digital oscillators has been suggested as a building block for a digital secure-communications system [13].

2. The two oscillator case

In the case of two oscillators, the continuous model (1) may be easily analyzed by defining the phase-difference ϕ as [9,6]

$$\phi = \theta_2 - \theta_1 \tag{3}$$

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whereupon Eq. (1) give

$$\frac{d}{dt}\phi = \omega_2 - \omega_1 - 2\alpha \sin\phi \tag{4}$$

Equilibria of Eq. (4) are known as *phase-locked* solutions and occur if and only if

$$|\omega_2 - \omega_1| \leqslant 2|\alpha| \tag{5}$$

In the case that Eq. (5) does not hold, the behavior of system (1) is described as *drift*.

In the case of the digital system (2), we find that the phase-difference (3) satisfies the equation:

$$\phi^{n+1} = \phi^n + \omega_2 - \omega_1 - 2\alpha\, \mathrm{sgn}(\phi^n) \tag{6}$$

Simulation of Eq. (2) for a system of two oscillators shows that if Eq. (5) is satisfied, Eq. (6) exhibits a periodic solution with period T called a $T-$ cycle (see Table 1). Note that some entries involve two periods. Each period corresponds to a different attractive steady state, each of which has its own basin of attraction.

To illustrate this we choose the case that $\alpha = 1$ and $\Omega = \omega_2 - \omega_1 = 0$, i.e., both oscillators have the same uncoupled frequency (see Figs. 1 and 2). Here we have chosen both oscillators to have a frequency of $\omega_i = 0$, which is equivalent to imagining that we are attached to a coordinate system which rotates with oscillator 1, thus making $\omega_1 = 0$. Since the frequency of both oscillators is zero, the movement of the oscillators will be due only to the coupling between them. In one time interval, the oscillator that is ahead will move back one segment and the oscillator that is behind will move ahead one. If, initially, the oscillators are an odd number of segments apart, as in Fig. 1, then they will move closer together until they are on consecutive segments, thus exhibiting a 2-cycle, switching places each time interval. If, instead, the oscillators are an even number

Table 1
Periods of periodic motions obtained in the case of two oscillators with coupling coefficient α and various $\Omega = \omega_2 - \omega_1$. The symbol ∞ represents drift. Multiple entries correspond to different attractive steady states, each corresponding to different initial conditions. Results obtained by computer simulation.

	3	1, 2	6	3	2, 4	3, 6	6	1	∞
α	2	1, 2	4	2, 4	4	1	∞	∞	∞
	1	1, 2	2	1	∞	∞	∞	∞	∞
	0	1	∞	∞	∞	∞	∞	∞	∞
		0	1	2	3	4	5	6	7
						Ω			

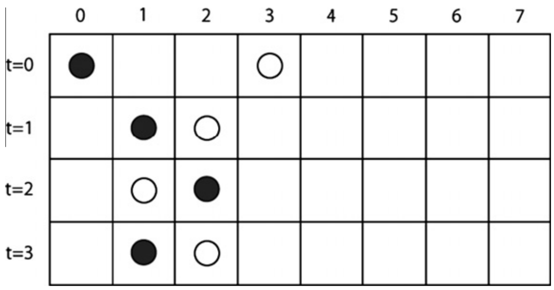

Fig. 1. Here both oscillators have frequency $\omega_i = 0$. The steady state depends on the initial conditions (cf. Fig. 2).

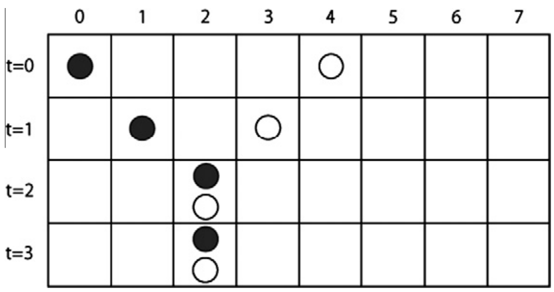

Fig. 2. Here both oscillators have frequency $\omega_i = 0$. The steady state depends on the initial conditions (cf. Fig. 1).

of segments apart, as shown in Fig. 2, then they will move closer together until they eventually land on the same segment and remain in place, thus exhibiting a 1-cycle.

3. The three oscillator case

In the case of three oscillators with $\alpha = 1$, we obtain Table 2. We see that the dynamics becomes more complex as the number of oscillators increases and that even 6-cycles are possible without increasing the coupling coefficient, see Fig. 3.

4. The *N*-oscillator case

In investigating the dynamics of N digital oscillators, we define ϕ_i as the position of oscillator i relative to that of oscillator 1:

$$\phi_i = \theta_i - \theta_1 \tag{7}$$

Note that $\phi_1 = 0$. In this case Eq. (2) give

$$\phi_i^{n+1} = \phi_i^n + \omega_i - \omega_1 + \alpha \sum_{k=1}^{N} (\mathrm{sgn}(\phi_k^n - \phi_i^n) - \mathrm{sgn}\phi_k^n), \quad i = 2, 3, \ldots N. \tag{8}$$

The following Theorem relates the distance a periodic T-cycle moves after one period of T time intervals.

Theorem. *Suppose there are N oscillators ($N \geqslant 2$) and that they are T-periodic with respect to Eq. (8). Define K to be the distance moved by the oscillators after one period. Then*

$$K = \frac{T}{N} \sum_{i=1}^{N} \omega_i \tag{9}$$

That is, K equals the period times the average uncoupled frequency.

Table 2
Periods of periodic motions obtained in the case of three oscillators with coupling coefficient $\alpha = 1$, for various frequency differences $\Omega_2 = \omega_2 - \omega_1$, $\Omega_3 = \omega_3 - \omega_1$. The symbol ∞ represents drift. Multiple entries correspond to different attractive steady states, each corresponding to different initial conditions. Note symmetry involving an invariance under the interchanging of Ω_2 and Ω_3. Results obtained by computer simulation.

	5	∞	∞	∞	∞	∞	∞
	4	∞	∞	1	∞	∞	∞
	3	1, 2	6	6	1, 2	∞	∞
Ω_3	2	3, 6	2, 4	3, 6	6	1	∞
	1	3, 6	3, 6	2, 4	6	∞	∞
	0	1, 2	3, 6	3, 6	1, 2	∞	∞
		0	1	2	3	4	5
				Ω_2			

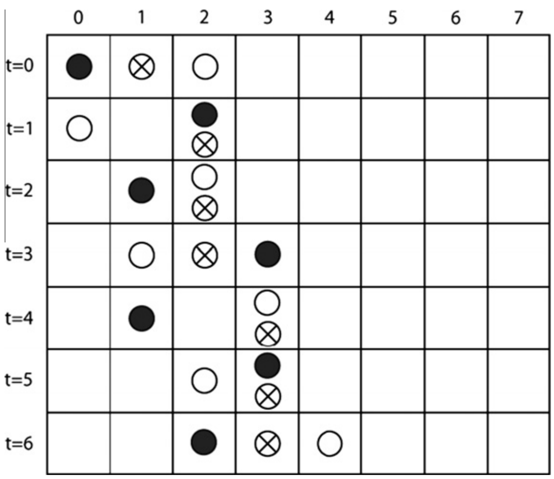

Fig. 3. A 6-cycle resulting from a three oscillator system with frequencies $\omega_1 = 0$ (black), $\omega_2 = 0$ (white), $\omega_3 = 1$ (cross).

As an example, take the system shown in Fig. 3. There the number of oscillators $N = 3$, the period of the motion is $T = 6$, and the average frequency of the oscillators is 1/3, giving the phase distance traveled in one period as $K = 2$, in agreement with Fig. 3.

We begin the proof by writing Eq. (2) with $n = T$.

$$\theta_i^{T+1} = \theta_i^T + \omega_i + \alpha \sum_{k=1}^{N} \operatorname{sgn}(\theta_k^T - \theta_i^T) \tag{10}$$

Define, $S_i^n = \alpha \sum_{k=1}^{N} \operatorname{sgn}(\theta_k^n - \theta_i^n)$. Then

$$\theta_i^{T+1} = \theta_i^T + \omega_i + S_i^T \tag{11}$$

$$= \theta_i^{T-1} + 2\omega_i + S_i^{T-1} + S_i^T \tag{12}$$

$$= \cdots \tag{13}$$

$$= \theta_i^1 + T\omega_i + \sum_{j=0}^{T-1} S_i^{T-j}. \tag{14}$$

Moving θ_i^1 to the left hand side of Eq. (14) and summing over i, we have

$$\sum_{i=1}^{N} (\theta_i^{T+1} - \theta_i^1) = T \sum_{i=1}^{N} \omega_i + \sum_{i=1}^{N} \sum_{j=0}^{T-1} S_i^{T-j}. \tag{15}$$

Changing the order of summation in the last term of Eq. (15) and noting that $\sum_{i=1}^{N} S_i^n = 0$ gives

$$\sum_{i=1}^{N} (\theta_i^{T+1} - \theta_i^1) = T \sum_{i=1}^{N} \omega_i \tag{16}$$

Since we have a period T motion, $\theta_i^{T+1} - \theta_i^1$, being the distance traveled by θ_i after one period, is the same for all oscillators i, and thus equal to K. Therefore

$$NK = T \sum_{i=1}^{N} \omega_i \tag{17}$$

$$K = \frac{T}{N} \sum_{i=1}^{N} \omega_i \tag{18}$$

5. Discussion

The system investigated in this work, Eq. (2), is an integer-valued version of the real-valued system (1). The question arises, how do familiar dynamical concepts such as stability, bifurcation and chaos, which occur in typical nonlinear dynamical systems such as (1), apply to the integer-valued system (2)?

To begin with, we note that the system (2) is nonlinear. The sine function, which is the source of nonlinearity in system (1), has been replaced in (2) by the sgn function, which is also nonlinear.

A related difference between systems (1) and (2) is the topology of the phase space. For system (2) it is N copies of the integers Z, while for system (1), the phase space is an N dimensional torus (because sine is a periodic function).

As shown in Tables 1 and 2, the system (2), when written in the form (8), exhibits a variety of periodic motions. By comparison with system (1), we are led to ask if these motions are stable? The answer is that traditional definitions of stability must be re-examined for integer-valued systems such as (2). Most stability definitions, such as Lyapunov stability, orbital stability and structural stability [6] involve comparing the behavior of the given motion with that of a slightly displaced motion, i.e., with a motion which lies in a neighborhood of the given motion. The problem with system (2) is that it lacks the notion of a *neighborhood*.

Nevertheless Figs. 1 and 2 show periodic motions which are *attractive*, a property which is often associated with stability. Therefore we propose the following definitions:

A periodic motion M in Eq. (8):

- is said to be *attractive* if there exists an initial condition which does not lie on M and which generates a motion which lands on M in finite time.
- is said to be *isolated* if it is not attractive, i.e., if the only motions which land on M in finite time are motions which start on M.

As an example of an isolated periodic motion, consider the system with $N = 2$ oscillators, coupling constant $\alpha = 1$, and frequencies $\omega_1 = 0$ and $\omega_2 = 2$. Suppose that at time $n = 0$ oscillator 1 is located at phase p and oscillator 2 is at phase $q > p + 3$. Then at time $n = 1$, oscillator 1 will be advanced to position $p + 1$ by the coupling term, whereas oscillator 2, which would be located at $q + 2$ in the absence of coupling, is moved back to $q + 1$ by the coupling term. Thus the phase difference between oscillator 1 and oscillator 2 remains constant and we have a phase-locked motion with period $T = 1$. However since p and $q > p + 3$ are arbitrary, a portion of the phase space is filled with periodic motions, and every initial condition therein lies on one such periodic motion. Each of these motions is therefore *isolated*. This is reminiscent of the phase portrait of a conservative oscillator in two dimensions, for example $x' = y$ and $y' = -x$, where the phase space is filled with periodic motions, none of which are attractive.

We define the *diameter* D of a configuration of N oscillators as the maximum distance between any two of them:

$$D = max\{|\theta_i - \theta_j|, \quad i,j = 1, 2, \ldots, N\} \tag{19}$$

A motion is said to *drift* if the diameter D approaches ∞ as time n approaches ∞.

Bifurcations, which can occur in system (1), involve qualitative changes in the system's dynamical behavior due to continuous changes in the parameters. Since all the parameters in system (2) are integers, a continuous change in parameters is not possible, and we cannot speak of bifurcations in system (2).

Can chaos occur in system (2)? A common criterion for chaos is sensitive dependence on initial conditions [9]. That is, one compares the motion resulting from two slightly different initial conditions, and asks if they continue to remain close to one another as time t approaches ∞. This concept cannot be applied to system (2) because initial conditions cannot be chosen arbitrarily close to one another.

Finally, we note that non-drifting aperiodic motions cannot occur in system (8). This is because if the motion is confined to a finite region of phase space (as in a non-drifting motion), then eventually every point in this region will have been exhausted and some point will have to be repeated, thus leading to a periodic motion.

6. Conclusion

We have proposed a new model for coupled digital oscillators, Eq. (2), which is a generalization of a well-known continuous-time model of coupled phase-only oscillators, Eq. (1). It has been shown that the new model exhibits two types of motions, those which involve periodic phase differences, and those which involve drift. In the case of the former, Tables 1 and 2 show a variety of attractive periodic motions, the period of which depends on the uncoupled oscillator frequencies ω_i and the coupling parameter α. For some parameters there exist multiple attractive periodic motions, each with its own basin of attraction. This work is expected to have application in digital technology.

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