Commun Nonlinear Sci Numer Simulat 15 (2010) 1980–1988

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Dynamics of a model of two delay-coupled relaxation oscillators

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ARTICLE INFO

Article history:
Received 26 August 2009
Accepted 4 September 2009
Available online 10 September 2009

Keywords:
Coupled oscillators
Devil's Staircase
Delay-differential equations

ABSTRACT

This paper investigates the dynamics of a new model of two coupled relaxation oscillators. The model replaces the usual DDE (differential-delay equation) formulation with a discrete-time approach with jumps. Existence, bifurcation and stability of in-phase periodic motions is studied. Simple periodic motions, which involve exactly two jumps per period, are found to have large plateaus in parameter space. These plateaus are separated by regions of complicated dynamics, reminiscent of the Devil's Staircase. Stability of motions in the in-phase manifold are contrasted with stability of motions in the full phase space.

1. Introduction

The dynamics of two coupled limit cycle oscillators has a history going back to the 1980s. Early research in this area involved systems without delay with both nearly sinusoidal oscillators [3,4,7] as well as relaxation oscillators [1,8,9,11]. More recent work has included delay in systems of nearly sinusoidal oscillators [12,13] and in relaxation oscillators [6].

The present work involves delay in a system of two coupled relaxation oscillators, in which the oscillators are modeled as having jumps. A model with jumps which has been used previously to study relaxation oscillations, is based on the simple first order differential equation [2,9,10]:

$$x' = -\beta x \tag{1}$$

together with the jump conditions:

$$x = \pm 1 \quad \text{jumps to} \quad x = \mp 2 \tag{2}$$

See Fig. 1.

We shall be interested in a system of two such oscillators, coupled by delay coupling:

$$x(t)' = -\beta x(t) - \alpha y(t - T) \tag{3}$$
$$y(t)' = -\beta y(t) - \alpha x(t - T) \tag{4}$$

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doi:10.1016/j.cnsns.2009.09.001

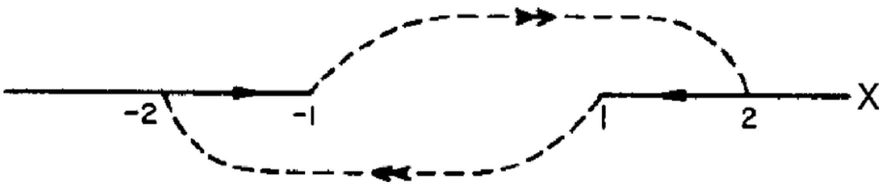

Fig. 1. Schematic diagram of model of relaxation oscillator. Single arrows indicate slow segments of the limit cycle while double arrows indicate rapid flow which is modeled as an instantaneous jump. We omit the region of phase space which lies between $x = -1$ and $x = 1$.

together with the associated jump conditions:

$$x = \pm 1 \quad \text{jumps to} \quad x = \mp 2 \tag{5}$$
$$y = \pm 1 \quad \text{jumps to} \quad y = \mp 2 \tag{6}$$

In particular we are interested in the limiting case in which $\alpha \gg \beta$, in which cases (3) and (4) become:

$$x(t)' = -\alpha y(t - T) \tag{7}$$
$$y(t)' = -\alpha x(t - T) \tag{8}$$

In this work, we shall be interested in replacing the differential equations (7) and (8) by iterative equations obtained by using finite differences:

$$\frac{x_{n+1} - x_n}{h} = -\alpha y_{n-N} \tag{9}$$
$$\frac{y_{n+1} - y_n}{h} = -\alpha x_{n-N} \tag{10}$$

where h is a step size and where the delay $T = Nh$. That is,

$$x_{n+1} = x_n - \alpha h y_{n-N} \tag{11}$$
$$y_{n+1} = y_n - \alpha h x_{n-N} \tag{12}$$

We interpret the jump conditions (5) and (6) by using the following rules:

$$\text{if} \quad x_n > 1 \quad \text{and} \quad x_{n+1} < 1 \quad \text{then} \quad x_{n+1} = -2 \tag{13}$$
$$\text{if} \quad x_n < -1 \quad \text{and} \quad x_{n+1} > -1 \quad \text{then} \quad x_{n+1} = 2 \tag{14}$$
$$\text{if} \quad y_n > 1 \quad \text{and} \quad y_{n+1} < 1 \quad \text{then} \quad y_{n+1} = -2 \tag{15}$$
$$\text{if} \quad y_n < -1 \quad \text{and} \quad y_{n+1} > -1 \quad \text{then} \quad y_{n+1} = 2 \tag{16}$$

For simplicity we choose $N = 2$ and we take $k = \alpha h$ for convenience:

$$x_{n+1} = x_n - k y_{n-2} \tag{17}$$
$$y_{n+1} = y_n - k x_{n-2} \tag{18}$$

Eqs. (17) and (18) exhibit an invariant manifold, which we refer to as the "IP manifold" and has the form $x_n = y_n$. Motions in the IP manifold are governed by the equation,

$$f_{n+1} = f_n - k f_{n-2} \tag{19}$$

where $f_n = x_n = y_n$. Note that the condition $x_n = y_n$ implies that only three initial conditions (i.c.'s) need to be specified (f_{-2}, f_{-1}, f_0).

Amongst all possible motions in the IP manifold there may exist periodic motions. If such a periodic motion exists we will refer to it as the "IP mode". If the IP mode has period M, then

$$f_{M+j} = f_j \tag{20}$$

Since the IP mode is periodic it will involve a jump from -1 to 2. Thus, we can make one of the i.c.'s equal to 2 without any loss of generality. Therefore, an IP mode lives on the invariant manifold $x_n \equiv y_n$ and can be generated by an i.c. of the form $(f_{-2}, f_{-1}, 2)$ which involves two independent parameters.

2. System with no jumps

A motion in the system with no jumps is determined by six i.c.'s denoted by, $(x_{-2}, x_{-1}, x_0) \times (y_{-2}, y_{-1}, y_0)$. By specifying these values x_n and y_n can be iterated forward for any n. Eqs. (17) and (18) without jumps exhibit a fixed point at the origin,

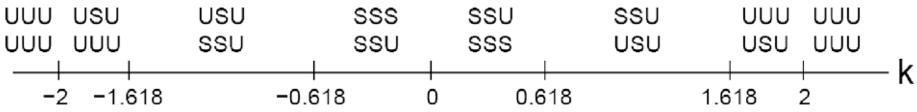

Fig. 2. The stability of the system with no jump conditions is summarized here. The system is six dimensional and thus has six corresponding eigenvalues which satisfy Eq. (22). The letter S (stable) signifies that an eigenvalue satisfies $|\lambda| < 1$, while the letter U (unstable) signifies that $|\lambda| > 1$. For the system to be stable all eigenvalues must satisfy $|\lambda| < 1$. If any eigenvalue satisfies $|\lambda| > 1$ then the system is unstable. The figure shows that the system is unstable for all values of k.

$x_n = y_n = 0$. To determine the stability of the origin we set $x_n = A\lambda^n,\ y_n = B\lambda^n$, which when substituted into Eqs. (17) and (18) give

$$\begin{pmatrix} \lambda - 1 & k\lambda^{-2} \\ k\lambda^{-2} & \lambda - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{21}$$

For a nontrivial solution, we require the determinant to vanish, giving the following conditions:

$$\lambda^3 - \lambda^2 - k = 0 \quad \text{or} \quad \lambda^3 - \lambda^2 + k = 0 \tag{22}$$

For the transition between stable and unstable, we require the necessary condition $|\lambda| = 1$, i.e. $\lambda = e^{i\omega}$:

$$e^{3i\omega} - e^{2i\omega} \pm k = 0 \tag{23}$$

which gives

$$i(\sin 3\omega - \sin 2\omega) + (\cos 3\omega - \cos 2\omega \pm k) = 0 \tag{24}$$

That is, we may solve

$$\sin 3\omega - \sin 2\omega = 0 \tag{25}$$

for each of its real roots ω, and then compute the corresponding value of k from

$$\pm k = \cos 2\omega - \cos 3\omega \tag{26}$$

Eq. (25) becomes,

$$\sin 3\omega - \sin 2\omega = -\sin\omega(\sin^2\omega - 3\cos^2\omega + 2\cos\omega) \tag{27}$$

$$= \sin\omega(-4\cos^2\omega + 2\cos\omega + 1) = 0 \tag{28}$$

which gives that $\sin\omega = 0$ or $\cos\omega = \frac{\sqrt{5}\pm 1}{4}$. Eq. (26) becomes,

$$\pm k = \cos 2\omega - \cos 3\omega = -4\cos^3\omega + 2\cos^2\omega + 3\cos\omega - 1 \tag{29}$$

Thus $\sin\omega = 0$ gives $k = 0$ and $k = \pm 2$, while $\cos\omega = \frac{\sqrt{5}\pm 1}{4}$ gives $k = \pm\frac{\sqrt{5}+1}{2} = \pm 1.618\ldots$ and $k = \pm\frac{\sqrt{5}-1}{2} = \pm 0.618\ldots$. See Fig. 2, which displays these critical values of k, each corresponding to a point where the origin potentially changes stability. The system requires six i.c's and thus has six corresponding eigenvalues, which satisfy Eq. (22). For the system to be stable all eigenvalues must satisfy $|\lambda| < 1$. If any eigenvalue satisfies $|\lambda| > 1$ then the system is unstable. Thus we see that the origin is unstable for all values of k in the system with time delay, but no jumps.

3. System with jumps

Based on the jump conditions in Eqs. (13)–(16), we may add jumps to the evolution equation (19) by using the rule:

$$\text{if} \quad f_n > 1 \quad \text{and} \quad f_{n+1} < 1 \quad \text{then} \quad f_{n+1} = -2 \tag{30}$$

$$\text{if} \quad f_n < -1 \quad \text{and} \quad f_{n+1} > -1 \quad \text{then} \quad f_{n+1} = 2 \tag{31}$$

Numerical simulation of the resulting system shows that the IP manifold can support stable periodic motions. For example, for $k = 0.6$ we observe a stable period-10 motion. See Fig. 3.

In the system governed by Eqs. (19), (30) and (31), a periodic motion is determined by specifying i.c.'s of the form $(f_{-2}, f_{-1}, 2)$. Here the jumps in the system have allowed one of the i.c.'s to be set to 2. In a sense, the jumps work as a many-to-1 transformation where points of the form (f_{-2}, f_{-1}, f_0) get mapped to $(f_{-2}, f_{-1}, 2)$ whenever $f_{-1} < -1$ and $f_0 > -1$. That is, all parameters satisfying $f_0 > -1$ get mapped to $f_0 = 2$ due to the jump. Thus only two i.c.'s need to be specified for any periodic motion, assuming the motion in question has two jumps.

The question arises as to which values of k correspond to a periodic motion of a given period M in the IP manifold. We offer the following analytical approach to this question. Since 2 is a "jump to" point we can always start a periodic motion with that i.c. For example, in a period-10 IP mode, as shown in Fig. 3, we may without loss of generality choose,

$$f_0 = 2 \tag{32}$$

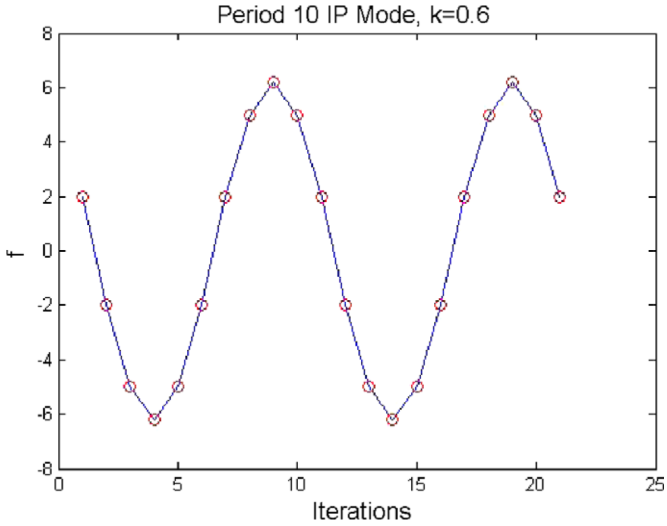

Fig. 3. Period-10 IP mode with $k = 0.6$.

The next point, f_1, satisfies Eq. (19) with $n = 0$:

$$f_1 = f_0 - kf_{-2} \tag{33}$$

Since the IP mode has been hypothesized to have period $M = 10$, then by Eq. (20) we have $f_{10-2} = f_{-2}$, i.e. $f_{-2} = f_8$ and,

$$f_1 = f_0 - kf_8 \tag{34}$$

Similarly we have for $f_2, \ldots, f_9$:

$$f_2 = f_1 - kf_9 \tag{35}$$
$$f_3 = f_2 - kf_0 = x_2 - 2k \tag{36}$$
$$f_4 = f_3 - kf_1 \tag{37}$$
$$f_5 = f_4 - kf_2 = -2 \tag{38}$$
$$f_6 = -2 - kf_3 = -f_1 \tag{39}$$
$$f_7 = f_6 - kf_4 = -f_2 \tag{40}$$
$$f_8 = f_7 + 2k = -f_3 \tag{41}$$
$$f_9 = f_8 - kf_6 = -f_4 \tag{42}$$

Here we have used the fact observed in Fig. 3 that the IP mode is "antisymmetric" with $f_{n+5} = -f_n$. In particular, this gives that the second jump occurs at f_5. Solving Eqs. (32)–(42) for $f_0 \ldots f_5$ gives the following results:

$$f_0 = 2 \tag{43}$$
$$f_1 = \frac{2}{1-k} \tag{44}$$
$$f_2 = \frac{2(k^2-k-1)}{k-1} \tag{45}$$
$$f_3 = \frac{2}{1-k} \tag{46}$$
$$f_4 = 2 \tag{47}$$
$$f_5 = \frac{2(k^3-k^2-2k+1)}{1-k} \tag{48}$$

For the IP mode to exist, the following six inequalities must be satisfied:

$$f_0 \ldots f_4 > 1, \quad f_5 = f_4 - kf_2 < 1 \tag{49}$$

A detailed analysis of these inequalities (omitted here for brevity) shows that all six will be satisfied if and only if $k \in [1 - \frac{1}{\sqrt{2}}, 1)$. Thus a period-10 IP mode only exists in this range of k.

If we follow the same process for periodic motions of period 3–9 we find that there is no k that satisfies our restrictions (given that only two jumps occur).

In the case of a period-11 IP mode, the jump to -2 may occur at either f_5 or f_6. Tentatively assuming that the jump occurs at f_6, we have

$$f_0 = 2 \tag{50}$$
$$f_1 = f_0 - kf_9 \tag{51}$$
$$f_2 = f_1 - kf_{10} \tag{52}$$
$$f_3 = f_2 - 2k \tag{53}$$
$$f_4 = f_3 - kf_1 \tag{54}$$
$$f_5 = f_4 - kf_2 \tag{55}$$
$$f_6 = f_5 - kf_3 = -2 \tag{56}$$
$$f_7 = -2 - kf_4 \tag{57}$$
$$f_8 = f_7 + 2f_5 \tag{58}$$
$$f_9 = f_8 + 2k \tag{59}$$
$$f_{10} = f_9 - kf_7 \tag{60}$$

This gives,

$$f_0 = 2 \tag{61}$$
$$f_1 = -\frac{2k^7 + 2k^4 + 6k^2 - 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{62}$$
$$f_2 = -\frac{2k^6 + 2k^5 - 4k^4 + 12k^3 + 6k^2 - 4k - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{63}$$
$$f_3 = -\frac{2k^8 + 2k^6 + 2k^5 + 6k^4 + 4k^3 + 6k^2 - 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{64}$$
$$f_4 = -\frac{2k^6 + 6k^4 - 2k^3 + 8k^2 - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{65}$$
$$f_5 = \frac{2k^7 - 4k^5 + 6k^4 + 8k^3 - 12k^2 - 2k + 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{66}$$
$$f_6 = \frac{2k^9 + 4k^7 + 2k^6 + 2k^5 + 10k^4 + 14k^3 - 14k^2 - 4k + 2}{k^7 + 5k^3 - 4k^2 + 1} = -2 \tag{67}$$
$$f_7 = \frac{6k^5 - 2k^4 - 2k^3 + 8k^2 - 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{68}$$
$$f_8 = -\frac{2k^8 - 4k^6 + 10k^4 - 10k^3 - 10k^2 + 4k + 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{69}$$
$$f_9 = \frac{4k^6 + 2k^3 + 10k^2 - 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{70}$$
$$f_{10} = -\frac{2k^6 - 2k^5 - 2k^4 + 6k^3 - 12k^2 + 2}{k^7 + 5k^3 - 4k^2 + 1} \tag{71}$$

Requiring that $f_0 \dots f_5 > 1$, $f_6 = f_5 - kf_3 < 1$ and $f_{11} = f_{10} - kf_8 > -1$ gives that the parameter k must be in the interval $(0.2397, 0.2628)$ for the system to exhibit a period-11 motion. Examination of Eqs. (19) and (30) shows that it is invariant with respect to the transformation $f \to -f$. Thus the IP mode with a jump at f_5 can obtained from the IP mode with jump at f_6 by a flip $(f \to -f)$ and a translation. Therefore the acceptable values of k are the same for both IP modes. The critical values of periodic motions up to period-20 can be found in Fig. 4.

As can be seen in Fig. 4, there are intervals of k where motions of a certain period exist. The size of the intervals for even number periods is significantly larger than those for odd number periods. IP modes with periods less than 10 do not exist. As the period increases the corresponding range of k becomes smaller.

Furthermore, there appear to be gaps between the parameter ranges of subsequent periods. Upon closer inspection of the parameter range between a period-10 motion and a period-11, we find that there exists periodic motions in this range with periods larger than 10 or 11. In fact, the IP modes in this parameter range are a combination of period-10 and period-11 motions. We will use the following notation to represent a typical periodic motion in this region:

$$(10_{m_1}, 11_{n_1}, 10_{m_2}, 11_{n_2}, 10_{m_3}, 11_{n_3}, \dots) \tag{72}$$

where 10_{m_1} represents m_1 period-10 segments and 11_{n_1} represents n_1 period-11 segments. For example, if we take $k = 0.28$ we find that the IP mode consists of a $(10_1, 11_1)$ motion, i.e. a period-10 motion followed by a period-11 to give a full period of 21. As another example, for $k = 0.2620$, we have a $(10_1, 11_7)$ motion, i.e. a period-10 followed by seven period-11 motions.

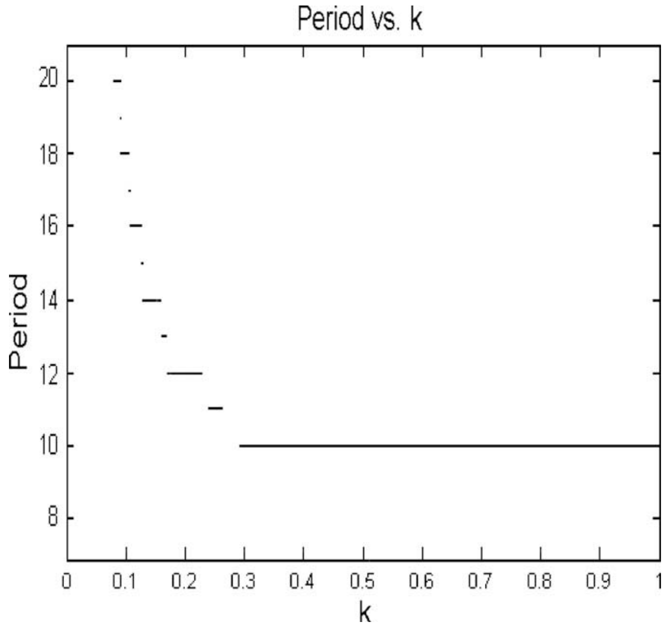

Fig. 4. The ranges of k for which IP modes of different periods exist.

If we consider the number of period-10 segments to the total number of segments, we see that the ratio increases as the value of k moves from the range of a period-11 to the range of a period-10. In essence, we have a Devil's Staircase. As k moves in the range between period-10 and period-11, the ratio of period-10 segments to the total number of segments will evidently go through every rational number less than 1. Table 1 gives the ratio and period for sample values of k. Similar behavior can be observed in the values of k between other subsequent period ranges.

4. Stability of the in phase mode

We wish to distinguish between two types of stabilities for the IP mode.

Type 1: Only i.c.'s of the form $x_n \equiv y_n$ are considered. All motions remain in the IP manifold.
Type 2: General i.c.'s are considered. Motions which begin off the IP manifold will not in general lie on it.

Table 1
The corresponding period and ratio of period-10 IP modes to total number of IP mode segments for values of k in the region between period-11 and period-10 IP modes. Results obtained by numerical simulation of Eqs. (19), (30) and (31).

k	Period segments	Ratio
0.2628	$10_1, 11_9$	1/10
0.2629–0.2631	$10_1, 11_6$	1/7
0.2632	$10_1, 11_5$	1/6
0.2633–0.2650	$10_1, 11_4$	1/5
0.2651–0.2652	$10_1, 11_3$	1/4
0.2653–0.2771	$10_1, 11_2$	1/3
0.2772–0.2777	$10_1, 11_2,\ 10_1, 11_1, 10_1,\ 11_1$	3/7
0.2778	$10_1, 11_1, 10_1,\ 11_1, 10_1, 11_1,\ 10_1, 11_1, 10_1,\ 11_2$	5/11
0.2779–0.2843	$10_1, 11_1$	1/2
0.2844–0.2845	$10_2, 11_1,\ 10_1, 11_1, 10_1,\ 11_1, 10_1, 11_1$	5/9
0.2846–0.2882	$10_2, 11_1, 10_1, 11_1$	3/5
0.2883–0.2903	$11_1, 10_2$	2/3
0.2904–0.2914	$10_3, 11_1,\ 10_2, 11_1$	5/7
0.2915–0.2921	$10_3, 11_1$	3/4
0.2922–0.2923	$10_1, 11_1,\ 10_3, 11_1, 10_3$	7/9
0.29250–0.2926	$11_1, 10_4$	4/5
0.2927	$11_1, 10_4, 11_1,\ 10_5$	9/11
0.2928	$11_1, 10_5$	5/6

We note that Type 2 stability implies Type 1 stability, since Type 1 is a special case of Type 2. In the case that we have Type 2 instability and Type 1 stability we say that the IP manifold is unstable, while the IP mode is stable in the IP manifold.

In the in-phase manifold we have that $x_n = y_n = f_n$, where (Eqs. (19), (30) and (31))

$$f_{n+1} = f_n - kf_{n-2} \tag{73}$$

$$\text{if} \quad f_n > 1 \quad \text{and} \quad f_{n+1} < 1 \quad \text{then} \quad f_{n+1} = -2 \tag{74}$$

$$\text{if} \quad f_n < -1 \quad \text{and} \quad f_{n+1} > -1 \quad \text{then} \quad f_{n+1} = 2 \tag{75}$$

For Type 2 stability we consider general i.c.'s of the form $(x_{-2}, x_{-1}, 2) \times (y_{-2}, y_{-1}, 2)$. Type 2 stability will tell us if a deviation off the IP manifold will result in motion that will move towards the IP manifold or away from it. Since Type 2 stability involves four i.c.'s, we may choose two of the i.c.'s to lie in the IP manifold and the other two to lie off the IP manifold. The two i.c.'s which lie in the IP manifold will determine Type 1 stability. Thus a Type 2 stability analysis will include a Type 1 stability analysis.

To determine Type 2 stability we add four deviations to the i.c.'s of our system in the IP mode. Since we fix x_0 and y_0 at 2, we need only to specify x_{-1}, x_{-2}, y_{-1} and y_{-2}.

Let us consider a deviation, $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, off the IP mode at values of k that correspond to a period-10 motion. Thus we have the following i.c.'s (cf. Eqs. (43)–(48)):

$$x_{-2} = \frac{-2}{1-k} + \epsilon_3 \tag{76}$$

$$x_{-1} = -2 + \epsilon_4 \tag{77}$$

$$x_0 = 2 \tag{78}$$

$$y_{-2} = \frac{-2}{1-k} + \epsilon_1 \tag{79}$$

$$y_{-1} = -2 + \epsilon_2 \tag{80}$$

$$y_0 = 2 \tag{81}$$

We assume that the deviations are small enough such that the jumps occur at the same time for x and y. Using Eqs. (17) and (18) and the jump conditions found in (13)–(16) we may use Eqs. (76)–(81) to generate values for $x_1, y_1, x_2, y_2, \ldots, x_{10}, y_{10}$. Assuming the jump to -2 occurs at x_5 and y_5, this results in

$$x_8 = -\frac{\epsilon_1 k^4 + (-2\epsilon_4 - 2\epsilon_3 - \epsilon_1)k^3 + (2\epsilon_4 + 2\epsilon_3)k^2 - 2}{k-1} \tag{82}$$

$$x_9 = (-\epsilon_2 - 2\epsilon_1)k^3 + (2\epsilon_4 + 2\epsilon_3)k^2 - 2 \tag{83}$$

$$x_{10} = 2 \tag{84}$$

$$y_8 = -\frac{\epsilon_3 k^4 + (-\epsilon_3 - 2\epsilon_2 - 2\epsilon_1)k^3 + (2\epsilon_2 + 2\epsilon_1)k^2 - 2}{k-1} \tag{85}$$

$$y_9 = (-\epsilon_4 - 2\epsilon_3)k^3 + (2\epsilon_2 + 2\epsilon_1)k^2 - 2 \tag{86}$$

$$y_{10} = 2 \tag{87}$$

Taking the difference between Eqs. (82)–(87) and these same equations with $\epsilon_i = 0$, we obtain the following dynamic on the deviations:

$$\varepsilon' = \begin{bmatrix} -k^3 & 0 & 2k^2 & 2k^2 \\ -2k^3 & -k^3 & 2k^2 & 2k^2 \\ 2k^2 & 2k^2 & -k^3 & 0 \\ 2k^2 & 2k^2 & -2k^3 & -k^3 \end{bmatrix} \varepsilon \tag{88}$$

where $\varepsilon' = [\,\epsilon_1' \quad \epsilon_2' \quad \epsilon_3' \quad \epsilon_4'\,]$. The eigenvalues for the above matrix are,

$$\lambda_{1,2} = -k^3 \mp 2\sqrt{1-k}\,k^2 + 2k^2 \tag{89}$$

$$\lambda_{3,4} = -k^3 \mp 2k^2\sqrt{k+1} - 2k^2 \tag{90}$$

where $\lambda_{1,2}$ corresponds to Type 1 stability. (Type 2 stability is associated with all four eigenvalues.)

For stability we require that all eigenvalues $|\lambda| < 1$. This turns out to give the following conditions for stability:

For deviations normal to IP manifold, stability requires

$$k \leqslant \frac{\left(\sqrt{59} + 3\sqrt{3}\right)^{\frac{2}{3}} - 2\,2^{\frac{2}{3}}}{2^{\frac{1}{3}}\sqrt{3}\left(\sqrt{59} + 3\sqrt{3}\right)^{\frac{1}{3}}} \approx 0.4534 \tag{91}$$

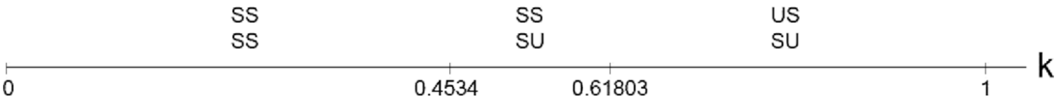

Fig. 5. Stability of the $M = 10$ IP mode. The letter S (stable) signifies that an eigenvalue satisfies $|\lambda| < 1$, while the letter U (unstable) signifies that $|\lambda| > 1$. For the system to be Type 2 stable all eigenvalues must satisfy $|\lambda| < 1$. If any eigenvalue satisfies $|\lambda| > 1$ then the system is Type 2 unstable. In each set of four letters, the top two letters correspond to deviations parallel to the IP manifold and thus to Type 1 stability. For $k < 0.4534$ the period-10 IP mode is Type 2 stable, whereas for $0.4534 < k < 0.61803$ it is Type 2 unstable but Type 1 stable. For $k > 0.61803$ it is both Type 2 and Type 1 unstable.

For deviations parallel to IP manifold, stability requires

$$k \leqslant \frac{\sqrt{5}-1}{2} \approx 0.61803 \tag{92}$$

See Fig. 5.

Likewise, we can consider a deviation off a period-11 motion. Thus we have the following i.c.'s (cf. Eqs. (61)–(71)):

$$x_{-2} = \frac{2k^6 + 6k^4 - 2k^3 + 8k^2 - 2}{k^7 + 5k^3 - 4k^2 + 1} + \epsilon_3 \tag{93}$$

$$x_{-1} = -\frac{2k^7 - 4k^5 + 6k^4 + 8k^3 - 12k^2 - 2k + 2}{k^7 + 5k^3 - 4k^2 + 1} + \epsilon_4 \tag{94}$$

$$x_0 = 2 \tag{95}$$

$$y_{-2} = \frac{2k^6 + 6k^4 - 2k^3 + 8k^2 - 2}{k^7 + 5k^3 - 4k^2 + 1} + \epsilon_1 \tag{96}$$

$$y_{-1} = -\frac{2k^7 - 4k^5 + 6k^4 + 8k^3 - 12k^2 - 2k + 2}{k^7 + 5k^3 - 4k^2 + 1} + \epsilon_2 \tag{97}$$

$$y_0 = 2 \tag{98}$$

After 11 iterations (one period),

$$\begin{aligned} x_9 = {} & \frac{(\epsilon_2 + 2\epsilon_1)k^{10} + (-2\epsilon_4 - 2\epsilon_3)k^9 + (5\epsilon_2 + 10\epsilon_1 - 2)k^6 + (-10\epsilon_4 - 10\epsilon_3 - 4\epsilon_2 - 8\epsilon_1)k^5}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(8\epsilon_4 + 8\epsilon_3 - 6)k^4 + (\epsilon_2 + 2\epsilon_1 + 2)k^3 + (-2\epsilon_4 - 2\epsilon_3 - 8)k^2 + 2}{k^7 + 5k^3 - 4k^2 + 1} \end{aligned} \tag{99}$$

$$\begin{aligned} x_{10} = {} & \frac{\epsilon_3 k^{11} + (-3\epsilon_2 - 4\epsilon_1)k^{10} + (2\epsilon_4 + 2\epsilon_3)k^9 + (5\epsilon_3 - 2)k^7 + (-4\epsilon_3 - 15\epsilon_2 - 20\epsilon_1)k^6}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(10\epsilon_4 + 10\epsilon_3 + 12\epsilon_2 + 16\epsilon_1 + 4)k^5 + (-8\epsilon_4 - 7\epsilon_3 - 6)k^4 + (-3\epsilon_2 - 4\epsilon_1 - 8)k^3}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(2\epsilon_4 + 2\epsilon_3 + 12)k^2 + 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \end{aligned} \tag{100}$$

$$x_{11} = 2 \tag{101}$$

$$\begin{aligned} y_9 = {} & \frac{(\epsilon_4 + 2\epsilon_3)k^{10} + (-2\epsilon_2 - 2\epsilon_1)k^9 + (5\epsilon_4 + 10\epsilon_3 - 2)k^6 + (-4\epsilon_4 - 8\epsilon_3 - 10\epsilon_2 - 10\epsilon_1)k^5}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(8\epsilon_2 + 8\epsilon_1 - 6)k^4 + (\epsilon_4 + 2\epsilon_3 + 2)k^3 + (-2\epsilon_2 - 2\epsilon_1 - 8)k^2 + 2}{k^7 + 5k^3 - 4k^2 + 1} \end{aligned} \tag{102}$$

$$\begin{aligned} y_{10} = {} & \frac{\epsilon_1 k^{11} + (-3\epsilon_4 - 4\epsilon_3)k^{10} + (2\epsilon_2 + 2\epsilon_1)k^9 + (5\epsilon_1 - 2)k^7 + (-15\epsilon_4 - 20\epsilon_3 - 4\epsilon_1)k^6}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(12\epsilon_4 + 16\epsilon_3 + 10\epsilon_2 + 10\epsilon_1 + 4)k^5 + (-8\epsilon_2 - 7\epsilon_1 - 6)k^4 + (-3\epsilon_4 - 4\epsilon_3 - 8)k^3}{k^7 + 5k^3 - 4k^2 + 1} \\ & + \frac{(2\epsilon_2 + 2\epsilon_1 + 12)k^2 + 2k - 2}{k^7 + 5k^3 - 4k^2 + 1} \end{aligned} \tag{103}$$

$$y_{11} = 2 \tag{104}$$

Again, taking the difference between Eqs. (99)–(104) and these same equations with $\epsilon_i = 0$, we obtain

$$\varepsilon' = \begin{bmatrix} -2k^3 & -k^3 & 2k^2 & 2k^2 \\ -4k^3 & -3k^3 & k^4+2k^2 & 2k^2 \\ 2k^2 & 2k^2 & -2k^3 & -k^3 \\ k^4+2k^2 & 2k^2 & -4k^3 & -3k^3 \end{bmatrix} \varepsilon \tag{105}$$

with eigenvalues,

$$\lambda_{1,2} = \frac{-5k^3 - 4k^2 \mp k^2\sqrt{4k^3 + 25k^2 + 40k + 16}}{2} \tag{106}$$

$$\lambda_{3,4} = \frac{-5k^3 + 4k^2 \mp k^2\sqrt{-4k^3 + 25k^2 - 40k + 16}}{2} \tag{107}$$

Requiring that all eigenvalues $|\lambda| < 1$ demands that the following conditions be satisfied for stability:

$$k < 1.66173 \quad \text{and} \quad k < 0.4067328 \tag{108}$$

In order for a period-11 IP mode to exist we saw that $0.2397 < k < 0.2628$ (Fig. 4). Thus, Eq. (108) are automatically satisfied and the period-11 mode is always Type 2 stable whenever it exists. Similarly, for a period-12 mode the condition for stability is $k < 0.371506$ which is always satisfied for those values of k for which the period-12 mode exists.

5. Conclusions

We have investigated the dynamics of a new model of two coupled relaxation oscillators. The model replaces the usual DDE (differential-delay equation) formulation with a discrete-time approach with jumps given by Eqs. (13)–(18). Omission of the jumps produced a linear system in which all motions grew without bound. The addition of jumps introduced nonlinearity, resulting in stable periodic motions.

The system has a single parameter k, which, when changed, produces changes in the dynamics of the in-phase (IP) mode, as shown in Fig. 4. We find either simple periodic motions which involve exactly two jumps per period, and which have large plateaus in Fig. 4, or complicated periodic motions which involve many jumps per period, and which exist for very small ranges in k, see Table 1. This structure is reminiscent of the Devil's Staircase [5]. We note that IP modes do not exist for periods less than 10.

Period-10 motions possess a large plateau with two types of stability, Fig. 5. Type 1 stability involves motion in the invariant IP manifold, while Type 2 stability involves motion in the full space. Periodic motions of period larger than 10 are found to be Type 2 stable. We note that the linear stability analyses compare favorably with numerical simulation for small deviations, i.e. the stable IP modes may have small basins of attraction.

Acknowledgments

This work was partially supported under the CMS NSF Grant CMS-0600174 "Nonlinear Dynamics of Coupled MEMS Oscillators". Author R.E.R. wishes to thank the Sloan Foundation for their support.

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