Analysis of a simplified MEMS oscillator

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Summary. A simplified model of a MEMS oscillator is presented and studied using perturbation methods. The model omits damping but includes nonlinearity and dependence of light absorption on interferometric gap. Analytic results are shown to be in agreement with results of numerical integration.

Introduction

In a JMEMS paper of 2004, the following model of a MEMS oscillator was presented [1]:

$$\ddot{z} + \frac{1}{Q}(\dot{z} - D\dot{T}) + (1 + CT)(z - DT) + \beta(Z - DT)^3 = 0 \quad \text{and} \quad \dot{T} + BT = AP(\alpha + \gamma \sin^2 2\pi(z - z_0)) \tag{1}$$

Here z is the displacement of a mechanical oscillator and T is its temperature due to laser illumination. In the mechanical equation Q is the quality factor, C is the stiffness change due to temperature, D is the displacement due to temperature and β is the coefficient of the cubic nonlinearity. In the thermal equation the quantities α and γ represent the average and contrast of the absorption of laser power, P is the laser power, A and B represent the thermal mass and heat loss rate. The offset, z_0, models the equilibrium position of the oscillator with respect to the interference field created by the oscillator/gap/substrate stack. This sophisticated model, which includes effects of damping, stiffness change due to heating, periodic dependence of light absorption on interferometric gap, and nonlinearity, was shown to support limit cycle oscillations.

The present work was motivated by the question, what is the simplest version of the above MEMS oscillator which supports limit cycle oscillations?

Our candidate is the following system which omits damping and various other effects:

$$\ddot{z} + z = T \qquad \text{and} \qquad \dot{T} + T = z^2 - zz_0 \tag{2}$$

For simplicity, all constants have been taken equal to unity. Numerical integration shows that this system supports a limit cycle, see Fig.1 LEFT.

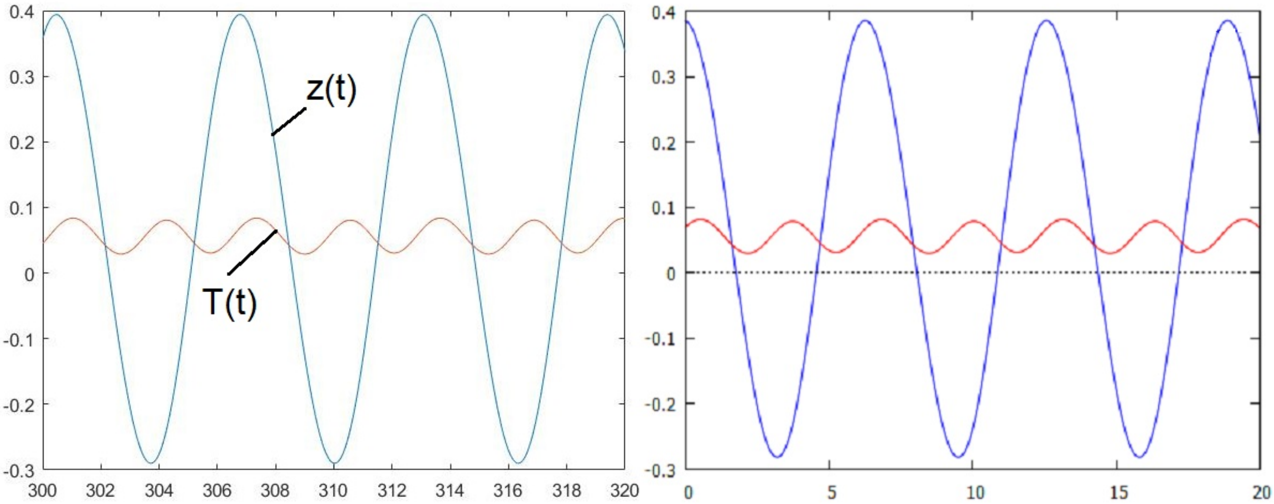

Figure 1: LEFT: Results of numerical integration of eqs.(2) for z_0=0.1. Note that the variable $T(t)$ appears to have twice the frequency of z(t). RIGHT:Eqs.(6) using eqs.(10)-(13) for $\varepsilon = 1$ and z_0=0.1.

Because of its relative simplicity, we hoped that this system could serve as a model for a system of coupled MEMS oscillators. By giving up some aspects of realistic modeling, the hope was that the dynamics of coupling, such as phase and frequency locking, could be more easily studied. As a prelude to such a study, we now present a perturbation analysis of eqs.(2).

Lindstedt's Method

In order to obtain an approximate analytic solution to eqs.(2), we will use Lindstedt's method. (See [2] for an introduction to Lindstedt's method.) The problem is how to scale the terms in eqs.(2). That is, where to insert the small parameter ε on which the perturbation method is based. After some experimentation, we decided on the following scheme:

$$\ddot{z} + z = T, \quad \text{and} \quad \dot{T} + T = \varepsilon z^2 - \varepsilon^2 z\, z_0 \tag{3}$$

We begin by stretching time with

$$\tau = \omega t, \quad \text{where} \quad \omega = 1 + k_1\varepsilon + k_2\varepsilon^2 + O(\varepsilon^3) \tag{4}$$

This gives:

$$\omega^2 z'' + z = T, \quad \text{and} \quad \omega T' + T = \varepsilon z^2 - \varepsilon^2 z\, z_0 \tag{5}$$

Next we expand as usual:

$$z = z_1 + z_2\varepsilon + z_3\varepsilon^2 + O(\varepsilon^3) \quad \text{and} \quad T = T_1 + T_2\varepsilon + T_3\varepsilon^2 + O(\varepsilon^3) \tag{6}$$

Substituting in eq.(5) and collecting terms, we get:

$$z_1'' + z_1 = T_1 \quad \text{and} \quad T_1' + T_1 = 0 \tag{7}$$

$$z_2'' + z_2 = -2k_1 z_1'' + T_2 \quad \text{and} \quad T_2' + T_2 = z_1^2 - k_1 T_1' \tag{8}$$

$$z_3'' + z_3 = -2k_1 z_2'' + (-2k_2 - k_1^2) z_1'' + T_3 \tag{9}$$

We take the solution to eqs.(7) to be

$$z_1 = A\cos\tau \quad \text{and} \quad T_1 = 0 \tag{10}$$

Substituting (10) into (8) and removing secular terms, turns out to give $k_1 = 0$ and:

$$z_2 = A^2\left(\frac{1}{2} - \frac{\sin 2\tau}{15} - \frac{\cos 2\tau}{30}\right) \quad \text{and} \quad T_2 = A^2\left(\frac{1}{2} + \frac{\sin 2\tau}{5} + \frac{\cos 2\tau}{10}\right) \tag{11}$$

Substituting (10),(11) into (9) and removing secular terms, turns out to give:

$$A = \frac{\sqrt{10}\sqrt{z_0}}{3} \quad \text{and} \quad k_2 = -\frac{z_0}{27} \tag{12}$$

Then solving eqs.(9) for z_3 and T_3 gives:

$$z_3 = \frac{\sqrt{10}\ (\sin 3\tau - \cos 3\tau)\ {z_0}^{3/2}}{1296} \quad \text{and} \quad T_3 = -\frac{\sqrt{10}\, z_0^{3/2}\ (\sin 3\tau - \cos 3\tau - 4\cos\tau)}{162} \tag{13}$$

Fig.1 RIGHT shows a plot of eqs.(6) using eqs.(10)-(13) for comparison with Fig.1 LEFT.

Conclusions

By drastically simplifying the MEMS model eqs.(1) with the system (2), we were able to obtain a closed form approximate analytic solution. In particular we derived an expression for limit cycle amplitude as a function of parameter z_0, eq.(12.1). It can be shown that the system (2) exhibits a Hopf bifurcation in the parameter z_0 at $z_0 = 0$. Future work will involve studying systems of coupled MEMS oscillators based on the system (2).

Acknowledgement

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References

[1] K.Aubin, M.Zalalutdinov, T.Alan, R.B.Reichenbach, R.Rand, A.Zehnder, J.Parpia and H.Craighead (2004) Limit Cycle Oscillations in CW Laser-Driven NEMS *J. Microelectromechanical Systems* **13**:1018-1026

[2] R.H.Rand (2012) "Lecture Notes in Nonlinear Vibrations" Published on-line by The Internet-First University Press http://ecommons.library.cornell.edu/handle/1813/28989