Duffing-Type Oscillators with Amplitude-Independent Period

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Abstract Nonlinear oscillators with hardening and softening cubic Duffing nonlinearity are considered. Such classical conservative oscillators are known to have an amplitude-dependent period. In this work, we design oscillators with the Duffing-type restoring force but an amplitude-independent period. We present their Lagrangians, equations of motion, conservation laws, as well as solutions for motion.

1 Introduction

Classical Duffing oscillators are governed by

$$\ddot{x} + x \pm x^3 = 0. \tag{1}$$

Their restoring force $F = x \pm x^3$ includes a linear geometric term as well as a cubic geometric term: a positive sign in front of the cubic term corresponds to a hardening Duffing oscillator (HDO) and the negative one to a softening Duffing oscillator (SDO) [4, 6]. Unlike the majority of nonlinear oscillators, both of them have a closed-form exact solution, which is expressed in terms of Jacobi cn or sn elliptic functions. These solutions corresponding to the following initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0, \tag{2}$$

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Table 1 Solutions for motion of the HDO and SDO (*in case of the SDO one requires $|A| < 1$ for the closed orbits around the stable origin)

	Solution for motion	Frequency	Elliptic modulus
HDO:	$x_{\mathrm{HDO}} = A\mathrm{cn}\left(\omega_{\mathrm{HDO}}t, k_{\mathrm{HDO}}\right)$	$\omega_{\mathrm{HDO}}^2 = 1 + A^2$	$k_{\mathrm{HDO}}^2 = \frac{A^2}{2\omega_{\mathrm{HDO}}^2} = \frac{A^2}{2(1+A^2)}$
SDO*:	$x_{\mathrm{SDO}} = A\mathrm{sn}\left(\omega_{\mathrm{SDO}}t, k_{\mathrm{SDO}}\right)$	$\omega_{\mathrm{SDO}}^2 = 1 - \frac{A^2}{2}$	$k_{\mathrm{SDO}}^2 = \frac{A^2}{2\omega_{\mathrm{SDO}}^2} = \frac{A^2}{2\left(1-\frac{A^2}{2}\right)}$

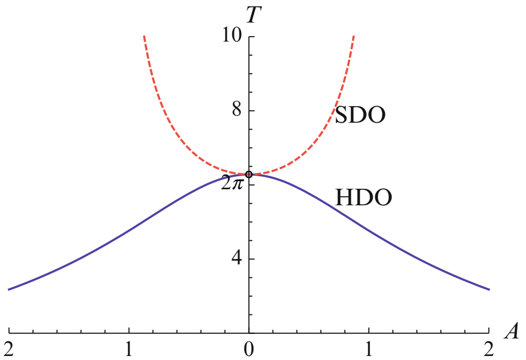

Fig. 1 Period of the HDO (*blue solid line*) and SDO (*red dashed line*) as a function of amplitude A

are given in Table 1 together with their frequencies ω and elliptic moduli k (see, for example, [4, 6] for more detail).

Given the fact that the period of cn and sn is $T = 4K(k)/\omega$, where $K(k)$ stands for the complete elliptic integral of the first kind, the following expressions are obtained for the period of the HDO and SDO:

$$T_{\mathrm{HDO}} = \frac{4K\left(\sqrt{\frac{A^2}{2(1+A^2)}}\right)}{\sqrt{1+A^2}}, \quad T_{\mathrm{SDO}} = \frac{4K\left(\sqrt{\frac{A^2}{2\left(1-\frac{A^2}{2}\right)}}\right)}{\sqrt{1-\frac{A^2}{2}}}. \tag{3}$$

As seen from these expressions and Fig. 1, the period of the HDO and SDO depends on the amplitude A. This leads us to the question of designing an oscillator having the Duffing-type restoring force $F = x \pm x^3$, but a constant, amplitude-independent period, corresponding to the so-called isochronous oscillators [1, 2]. In what follows we present such Duffing-type oscillators, both hardening and softening, modelled by

$$\ddot{x} + G(x, \dot{x}) + x \pm x^3 = 0, \tag{4}$$

finding also the corresponding Lagrangians, conservation laws, as well as solutions for motion.

2 Derivation

The derivation of the mathematical model (4) is based on the transformation approach [5], in which the kinetic energy E_k and potential energy E_p of nonlinear oscillators are made equal to the one of the simple harmonic oscillator (SHO)

$$E_{k\text{SHO}} = \frac{1}{2}\dot{X}^2, \quad E_{p\text{SHO}} = \frac{1}{2}X^2, \tag{5}$$

which is known to have a constant, amplitude-independent period (note that its generalized coordinate is labelled here by X).

2.1 Case I

To find the first mathematical model of the form (4), we assume that $E_p = E_p(x)$ and let $E_p \equiv E_{p\text{SHO}} = X^2/2$, obtaining

$$X = \sqrt{2E_p}. \tag{6}$$

Then, we also make the kinetic energy E_k of nonlinear oscillators equal to the one of the SHO and use Eq. (6) to derive

$$E_k = \frac{\dot{X}^2}{2} = \frac{\left(E_p'\right)^2}{4E_p}\dot{x}^2, \tag{7}$$

where $E_p' = dE_p/dx$. The differential equation of motion stemming from the Lagrangian $L = E_k - E_p$ has a general form

$$\ddot{x} + \left(\frac{E_p''}{E_p'} - \frac{E_p'}{2E_p}\right)\dot{x}^2 + \frac{2E_p}{E_p'} = 0. \tag{8}$$

The last term on the left-hand side $2E_p/E_p'$ is required to correspond to the Duffing restoring force $F=x \pm x^3$, which gives the potential energy

$$E_p = \frac{x^2}{2(1 \pm x^2)}. \tag{9}$$

Equation (7) now yields the kinetic energy

$$E_k = \frac{1}{2(1 \pm x^2)^3}\dot{x}^2, \tag{10}$$

so that Eq. (8) becomes

$$\ddot{x} \mp \frac{3x}{1 \pm x^2}\dot{x}^2 + x \pm x^3 = 0. \tag{11}$$

Based on $X^2/2 + \dot{X}^2/2 = const.$, the system exhibits the first integral

$$\frac{1}{(1 \pm x^2)^3}\dot{x}^2 + \frac{x^2}{1 \pm x^2} = \frac{A^2}{1 \pm A^2}. \tag{12}$$

Taking the solution for motion of the SHO in the form $X = a\cos(t + \alpha)$, where a and α are constants, and using Eqs. (6), (9) and (2), the solution for motion of Eq. (11) is obtained

$$\frac{A}{\sqrt{1 \pm A^2}}\cos t = \frac{x}{\sqrt{1 \pm x^2}}. \tag{13}$$

Numerical verifications of the analytical results for the motion (13) and phase trajectories (12) are shown in Figs. 2 and 3 for the HDO and SDO, respectively. These figures confirm that the analytical results coincide with the solutions obtained by solving the equation of motion (11) numerically. In addition, they illustrate the fact that the period stays constant despite the fact that the amplitude A changes, i.e., that the systems perform isochronous oscillations.

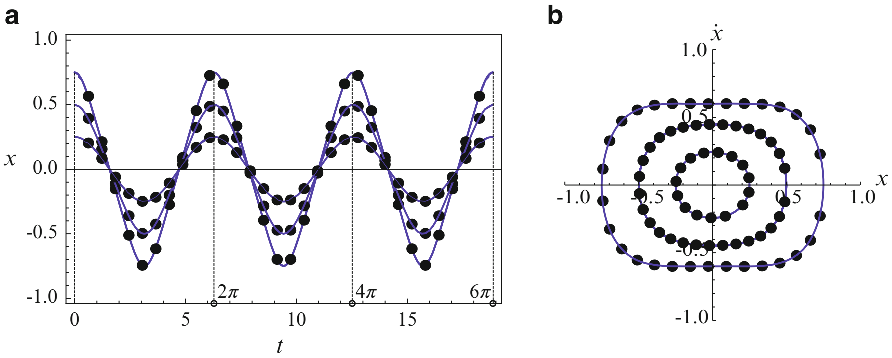

Fig. 2 Isochronous oscillations of the HDO, Eq. (11) for $A = 0.25$; 0.5; 0.75: (**a**) time histories obtained numerically from Eq. (11) (*black dots*) and from Eq. (13) (*blue solid line*); (**b**) phase trajectories obtained numerically from Eq. (11) (*black dots*) and from Eq. (12) (*blue solid line*) (upper signs are used in all these equations)

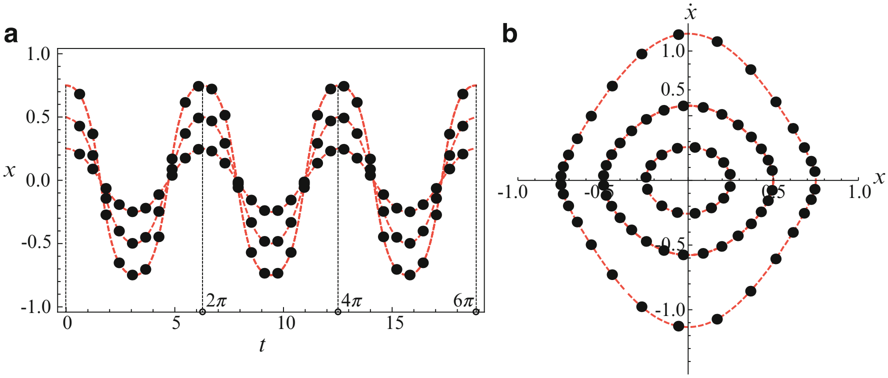

Fig. 3 Isochronous oscillations of the SDO, Eq. (11) for $A = 0.25; 0.5; 0.75$: (**a**) time histories obtained numerically from Eq. (11) (*black dots*) and from Eq. (13) (*red dashed line*); (**b**) phase trajectories obtained numerically from Eq. (11) (*black dots*) and from Eq. (12) (*red dashed line*) (lower signs are used in all these equations)

2.2 *Case II*

In this case we consider the system whose potential and kinetic energies are

$$E_p = \frac{1}{2}X^2 = \frac{1}{2}(x\,f)^2, \quad E_k = \frac{1}{2}\dot{X}^2 = \frac{1}{2}\left(\dot{x}\,f + x^2\,f\,'\right)^2, \tag{14}$$

where $f \equiv f(I)$, $I = \int_0^t x(t)\,dt$, and $f\,' = df/dI$.

The corresponding Lagrange's equation is

$$\ddot{x} + 3x\dot{x}\frac{f\,'}{f} + x + \frac{f\,''}{f}x^3 = 0. \tag{15}$$

This system has two independent first integrals. The first one is the energy conservation law stemming from $\dot{X}^2/2 + X^2/2 = const$:

$$\left(\dot{x}\,f + x^2\,f\,'\right)^2 + (x\,f)^2 = h_1, \quad h_1 = const. \tag{16}$$

The other first integral is related to the principle of conservation of momentum for the SHO $\dot{X} + \int_0^t X dt = \dot{X}(0) = const$. By using X and $\dot{X}$ from (14) and knowing that $dI/dt = x$, we obtain

$$\dot{x}\,f + x^2\,f\,' + \int f(I)\,dI = h_2, \quad h_2 = const. \tag{17}$$

In addition, as the solution for motion for the SHO can be written down as $X = a\sin(t+\alpha)$, the following should be satisfied:

$$a\sin(t+\alpha) = x\ f, \quad a\cos(t+\alpha) = \dot{x}\ f + x^2\ f\,'. \tag{18}$$

For the hardening-type nonlinearity in Eq. (15), one requires $f\,''/f = 1$, which leads to

$$f_{\mathrm{HDO}} = \exp\left(\int_0^t x(t)\,dt\right), \tag{19}$$

and the equation of motion takes the form

$$\ddot{x} + 3x\dot{x} + x + x^3 = 0. \tag{20}$$

Two first integrals (16) and (17) are

$$\exp(2x_1)\left[\left(x_3 + x_2^2\right)^2 + x_2^2\right] = h_1, \tag{21}$$

and

$$\exp(x_1)\left(x_3 + x_2^2 + 1\right) = h_2, \tag{22}$$

where

$$x_1 = \int_0^t x(t)\,dt, \quad x_2 = \dot{x}_1 = x, \quad x_3 = \dot{x}_2 = \dot{x}, \tag{23}$$

with initial conditions being [see Eq. (2)]

$$x_1(0) = 0, \quad x_2(0) = \dot{x}_1(0) = A, \quad x_3(0) = \ddot{x}_1(0) = 0. \tag{24}$$

Equation (21) is plotted in Fig. 4a for $h_1 = 1$. To analyze phase trajectories in more detail, Eq. (22) is squared and divided by Eq. (21) to obtain

$$\frac{\left(x_3 + x_2^2 + 1\right)^2}{\left(x_3 + x_2^2\right)^2 + x_2^2} = B, \quad B = const. \tag{25}$$

This expression agrees with the first integral obtained and studied in [3] and is plotted in Fig. 4b, where periodic solutions correspond to the case $B > 1$. Note that for the initial conditions (24), one has $B = 1 + 1/A^2$, which implies that B is always higher than unity.

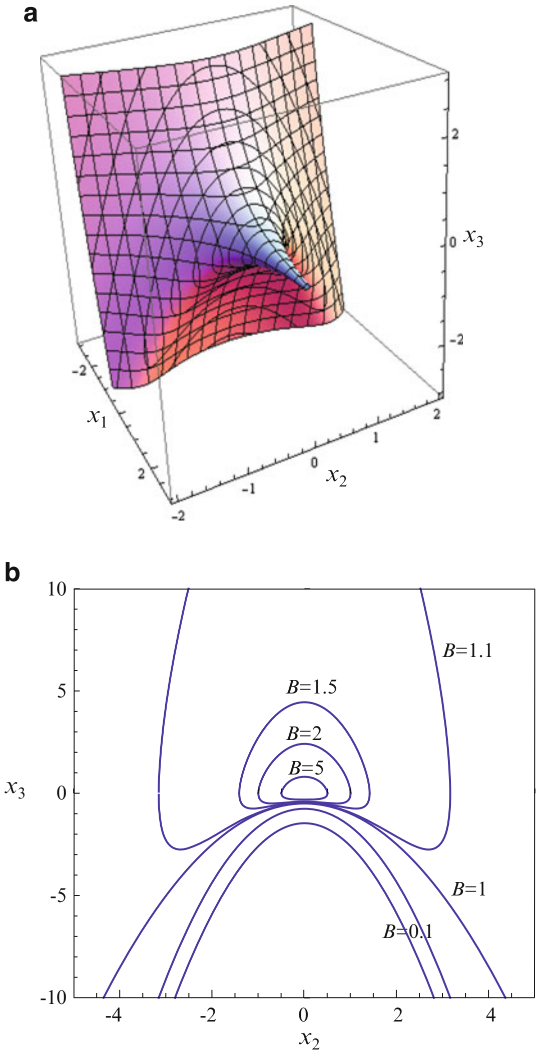

Fig. 4 (**a**) 3D plot of Eq. (21); (**b**) phase trajectories obtained from Eq. (25) for different values of B

By using (18) and (19) one can derive

$$\dot{x}\ \sin(t+\alpha) + x^2 \sin(t+\alpha) - x\cos(t+\alpha) = 0. \tag{26}$$

Its solution satisfying Eq. (2) is

$$x = \frac{\sin\left(t + \arctan\frac{1}{A}\right)}{\sqrt{1+\frac{1}{A^2}} - \cos\left(t + \arctan\frac{1}{A}\right)}. \tag{27}$$

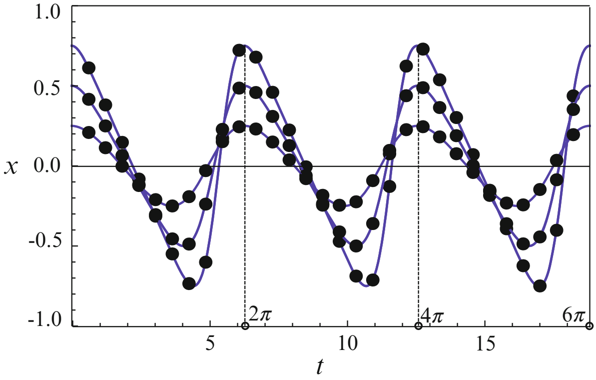

Fig. 5 Time response of the HDO, Eq. (20) for $A = 0.25; 0.5; 0.75$: numerically obtained solution from Eq. (28) (*black dots*) and from Eq. (27) (*blue solid line*)

To compare the analytical solution for motion (27) with a numerically obtained solution of the equation of motion (20), the latter is written down in the form

$$\dddot{x}_1 + 3\dot{x}_1\ddot{x}_1 + \dot{x}_1 + \dot{x}_1^3 = 0, \tag{28}$$

and numerically integrated by using the initial conditions (24). This comparison, plotted in Fig. 5, shows that these two types of solution are in full agreement as well as that the period is amplitude-independent.

The equation of motion (15) corresponds to the SDO if $f''/f = -1$, which is satisfied for

$$f_{\mathrm{SDO}} = \cos\left(\int_0^t x(t)\,dt\right). \tag{29}$$

This equation of motion is now given by

$$\ddot{x} - 3x\dot{x}\tan\left(\int_0^t x(t)\,dt\right) + x - x^3 = 0. \tag{30}$$

By using the notation given in Eq. (23), the equation of motion (30) transforms to

$$\dddot{x}_1 - 3\dot{x}_1\ddot{x}_1 \tan x_1 + \dot{x}_1 - \dot{x}_1^3 = 0, \tag{31}$$

with the initial conditions given in Eq. (24). Two first integrals (16) and (17) become

$$\left(x_3 \cos x_1 - x_2^2 \sin x_1\right)^2 + x_2^2 \cos^2 x_1 = h_1, \tag{32}$$

and

$$x_3 \cos x_1 - x_2^2 \sin x_1 + \sin x_1 = h_2. \tag{33}$$

These two integrals can be manipulated to exclude x_1 and to derive

$$\begin{aligned}(x_2^2 - 1)^2(h_1 + (-1 - h_1 + h_2^2)x_2^2 + x_2^4)^2 + 2(1 - h_1 - h_2^2 \\ +(-1 + h_1 - h_2^2)x_2^2)(h_1(x_2^2 - 1) - x_2^2(h_2^2 + x_2^2 - 1))x_3^2 \\ +((-1 + h_1)^2 - 2(1 + h_1)h_2^2 + h_2^4)x_3^4 = 0.\end{aligned} \tag{34}$$

For the initial conditions (24) one has $h_1 = A^2$ and $h_2 = 0$. Introducing these values into Eq. (34) and solving it with respect to x_3, the following explicit solution for phase trajectories is obtained:

$$x_3 = \pm(x_2^2 - 1)\sqrt{\frac{A^2 - x_2^2}{1 - A^2}}. \tag{35}$$

Combining equations in (18) and using $a = A$ and $\alpha = \pi/2$, we derive

$$\dot{x}\, A \cos t - x^2\sqrt{x^2 - A^2 \cos^2 t} + x A \sin t = 0. \tag{36}$$

Its solution satisfying Eq. (2) is

$$x = \frac{A \cos t}{\sqrt{1 - A^2 \sin^2 t}}. \tag{37}$$

This solution is plotted in Fig. 6 together with the numerical solution of Eq. (31) with Eq. (24) for different values of A. These solutions coincide and confirm isochronicity.

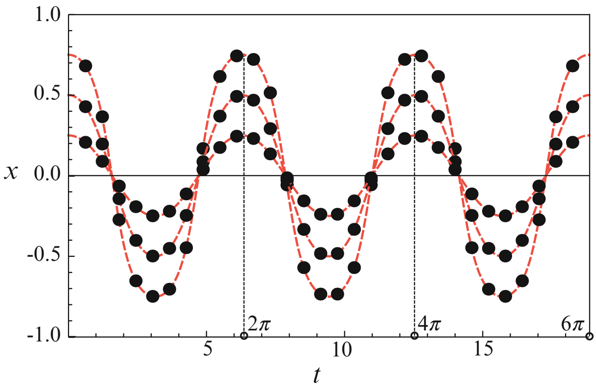

Fig. 6 Time response of the SDO, Eq. (30) with $A = 0.25; 0.5; 0.75$: numerically obtained solution from Eq. (31) (*black dots*) and from Eq. (37) (*red dashed line*)

3 Conclusions

In this work we have considered nonlinear oscillators with a hardening and softening Duffing restoring force. Unlike classical conservative Duffing oscillators, which have an amplitude-dependent period, the designed Duffing-type oscillators have the period that does not change with their amplitude and are, thus, isochronous. Two separate cases are considered with respect to the form of their potential and kinetic energy, which are made equal to the corresponding energies of the SHO, which is known to be isochronous. Corresponding equations of motions are derived, as well as their solutions for motion. Numerical verifications of these isochronous solutions are provided. In addition, two independent first integrals are presented: the energy-conservation law and the principle of conservation of momentum.

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References

1. Calogero, F.: Isochronous Systems. Oxford University Press, Oxford (2008)
2. Calogero, F.: Isochronous dynamical systems. Phil. Trans. R. Soc. A **369**, 1118–1136 (2011)
3. Iacono,R., Russo, F.: Class of solvable nonlinear oscillators with isochronous orbits. Phys. Rev. E **83**, 027601 (4 pages) (2011)
4. Kovacic, I., Brennan, M.J.: The Duffing Equation: Nonlinear Oscillators and their Behaviour. Wiley, Chichester (2011)
5. Kovacic, I., Rand, R.: About a class of nonlinear oscillators with amplitude-independent frequency. Nonlinear Dyn. **74**, 455–465 (2013)
6. Rand, R.H.: Lecture notes on nonlinear vibrations (version 53). http://dspace.library.cornell.edu/handle/1813/28989