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TWO PATHS TO ISOCHRONICITY IN A CLASS OF ONE DEGREE OF FREEDOM OSCILLATORS

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ABSTRACT

We propose a class of problems in nonlinear vibrations related to avoiding undesirable hysteresis and jump phenomena by designing an oscillator for which the backbone curve is a straight vertical line. In particular we consider the class of conservative oscillators of the form:

$$\ddot{x} + x\dot{x}^2 + f(x) = 0$$

and we choose $f(x)$ so that the frequency of oscillation is independent of amplitude. We do this in two ways:
1) by expanding $f(x)$ in a power series and using a perturbation method to compute the coefficients, and
2) by starting with a simple harmonic oscillator (which has a straight line backbone curve), and choosing a transformation which puts the resulting equation in the above form.
We show that both methods result in a closed form expression for $f(x)$ which involves the imaginary error function erfi(z).

INTRODUCTION

The simple harmonic oscillator (SHO)

$$\ddot{X} + X = 0 \tag{1}$$

is characterized by the property of isochronicity/isochrony [1], which means that it has a fixed, amplitude-independent frequency. Its backbone curve, which is a graphical presentation of the relationship between the amplitude of vibration R and the frequency ω is, thus, a straight vertical curve, as shown in Fig. 1.

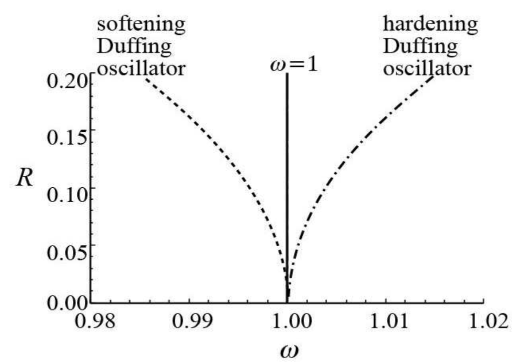

FIGURE 1. Numerically obtained backbone curves of the Duffing oscillator (2) (dotted and dashed-dotted lines) and a straight-line backbone curve (solid line).

However, if one keeps the system conservative and just adds/subtracts a cubic geometric term to/from Eq. (1), the dynamics of the corresponding so-called Duffing oscillator [2]- [4]

$$\ddot{x} + x \pm x^3 = 0 \tag{2}$$

completely changes and involves a certain relationship between the amplitude of vibration R of a typical periodic motion and its frequency ω. This relation is such that the backbone curve for Eq. (2) is bent to the right for the hardening Duffing oscillator (Eq. (2) with a plus sign in front of the cubic term) and to the left for the softening Duffing oscillator (Eq. (2) with a minus sign in front of the cubic term). When the Duffing oscillator is forced, this characteristic is known to cause hysteresis and jump phenomena (see, for example, [2], [3] or [4]).

There may be some situations where these phenomena are undesirable. This leads us to the question of designing a differential equation which is similar to the Duffing equation (2) in that it is conservative, but for which the backbone curve is a straight vertical line in the $\omega - R$ plane, as is in the case of the SHO (Fig. 1). To that end, we consider the class of conservative oscillators of the form:

$$\ddot{x} + x\dot{x}^2 + f(x) = 0 \tag{3}$$

and we choose $f(x)$ so that the frequency of oscillation is independent of amplitude. We do this in two ways:
1) by expanding $f(x)$ in a power series and using a perturbation method to compute the coefficients, and
2) by starting with the SHO (1) and choosing a transformation which puts the resulting equation in the above form.

This work is an extension of our previous papers [5], [6]. As discussed in [6], the system (3) is a special case of a class of systems studied by Sabatini [7] of the form:

$$\ddot{x} + p(x)\dot{x}^2 + q(x) = 0, \tag{4}$$

Another equation which has been shown to exhibit isochronicity is of the Lienard form:

$$\ddot{x} + u(x)\dot{x} + v(x) = 0 \tag{5}$$

The following authors have investigated isochronicity in Eq. (5): Sabatini [8], Iacono and Russo [9], Christopher and Devlin [10], Chandrasekar et al. [11], [12]. See Calogero [1] for an overview of isochronicity.

It should be mentioned that first results in this field are believed to date back to Galileo Galilei and Christian Huygens [13]. Although Galileo did not live to complete his design, he had believed that a pendulum is isochronous in the sense that the time it takes to complete one full swing is the same regardless of the size of the swing. Huygens, however, pushed this matter further, noting that this is true for pendulums that swing only a few degrees. He pursued the question of achieving perfect isochronicity and showed that it can be realized in a simple pendulum that wraps around the cycloid [14], [15].

In another paper [6], the present authors have shown that Eq. (3) has applications to systems which involve a mass dependent on position, or a kinematic mechanism with two masses and one degree of freedom.

POWER SERIES APPROACH

We begin by expanding $f(x)$ in Eq.(3) in the form:

$$f(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots \tag{6}$$

and look for the coefficients a_i in $f(x)$ such that Eq. (3), although nonlinear, has an amplitude-independent frequency.

To begin with, we note that Eqs. (3) and (6) correspond to a conservative system whose Lagrangian has the form

$$L = e^{x^2}\left(\frac{1}{2}\dot{x}^2 - g(x)\right) \tag{7}$$

where $g(x)$ is to be determined. The corresponding Lagrange's equation is:

$$\ddot{x} + x\dot{x}^2 + \frac{dg(x)}{dx} + 2xg(x) = 0 \tag{8}$$

Comparing Eqs. (3), (6) with (8) we see that $g(x)$ must satisfy the following equation:

$$\frac{dg(x)}{dx} + 2xg(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots \tag{9}$$

This may be solved for $g(x)$ by taking $g(x)$ in the form of a power series with even-powered terms:

$$g(x) = b_2x^2 + b_4x^4 + b_6x^6 + \cdots \tag{10}$$

Substitution of Eq. (10) into Eq. (9) leads to expressions for the b_i coefficients, the first few of which are:

$$b_2 = \frac{1}{2}, \quad b_4 = \frac{a_3 - 1}{4}, \quad b_6 = \frac{2a_5 - a_3 + 1}{12}, \quad \cdots \tag{11}$$

In order to use a perturbation method, we introduce a small parameter ε by setting $x = \sqrt{\varepsilon}\ \tilde{x}$ in Eqs. (3), (6) and drop the tilde for convenience, deriving:

$$\ddot{x} + x + \varepsilon x\dot{x}^2 + \varepsilon a_3 x^3 + \varepsilon^2 a_5 x^5 + \varepsilon^3 a_7 x^7 + \cdots = 0 \tag{12}$$

To obtain an approximate solution to Eq. (12), we expand x in a power series in ε:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{13}$$

Note that there is no need to expand frequency in a power series in ε as is usual in Lindstedt's method [4] as we are after a straight-line backbone curve.

Substituting Eq. (13) into Eq. (12) and collecting terms we get a sequence of equations, the first few of which are:

$$\ddot{x}_0 + x_0 = 0 \tag{14}$$

$$\ddot{x}_1 + x_1 = -x_0\dot{x}_0^2 - a_3 x_0^3 \tag{15}$$

$$\ddot{x}_2 + x_2 = -2\ x_0\dot{x}_0\dot{x}_1 - \dot{x}_0^2\ x_1 - 3\ a_3\ x_0^2\ x_1 -\ a_5\ x_0^5 \tag{16}$$

The solution to Eq. (14) is assumed to be

$$x_0 = R\cos t \tag{17}$$

whereupon Eq. (15) becomes:

$$\ddot{x}_1 + x_1 = \frac{a_3 - 1}{4} R^3 \cos 3t + \frac{3a_3 + 1}{4} R^3 \cos t \tag{18}$$

We take $a_3 = -1/3$ to remove resonance terms, and obtain

$$\ddot{x}_1 + x_1 = -\frac{1}{3} R^3 \cos 3t \tag{19}$$

which has the particular solution:

$$x_1 = -\frac{1}{24} R^3 \cos 3t \tag{20}$$

Substituting Eqs. (17), (20) into Eq. (16) yields

$$\ddot{x}_2 + x_2 = \frac{15a_5 - 1}{24} R^5 \cos t + \mathrm{NRT} \tag{21}$$

where NRT stands for non-resonant terms. For no resonance, we choose $a_5 = 1/15$. Proceeding in this way, we obtain the following values for the coefficients a_i in Eq. (6):

$$a_3 = -1/3 \tag{22}$$
$$a_5 = 1/15 = 1/(3*5) \tag{23}$$
$$a_7 = -1/105 = -1/(3*5*7) \tag{24}$$
$$a_9 = 1/945 = 1/(3^3*5*7) \tag{25}$$
$$a_{11} = -1/10395 = -1/(3^3*5*7*11) \tag{26}$$
$$a_{13} = 1/135135 = 1/(3^3*5*7*11*13) \tag{27}$$
$$a_{15} = -1/2027025 = -1/(3^4*5^2*7*11*13) \tag{28}$$
$$a_{17} = 1/34459425 = 1/(3^4*5^2*7*11*13*17) \tag{29}$$

Now it is important to note that the typical term in the foregoing list of coefficients may be written in the following compact form:

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!!}, \quad n = 1, 2, 3, \ldots \tag{30}$$

Thus, the straight-line backbone differential equation (3), (6), with the linear term included into the sum, can be written down as:

$$\ddot{x} + x\dot{x}^2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = 0 \tag{31}$$

It is remarkable that the sum in Eq. (31) can be represented in the following closed form:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = \sqrt{\frac{\pi}{2}}\, e^{-\frac{x^2}{2}}\ \mathrm{erfi}\left(\frac{x}{\sqrt{2}}\right) \tag{32}$$

where erfi denotes the "imaginary error function" defined as [16]

$$\mathrm{erfi}(z) = -i\ \mathrm{erf}(iz) \tag{33}$$

where erf represents the error function [16], [17],

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du \tag{34}$$

We note that $\operatorname{erfi}(z)$ satisfies the equation [16]:

$$\frac{d}{dz}\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \exp z^2 \tag{35}$$

Therefore, starting from a local perturbation analysis we have been able to obtain an expression for the straight-line backbone differential equation which is valid for all x, namely

$$\ddot{x} + x\dot{x}^2 + \sqrt{\frac{\pi}{2}}\, e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) = 0 \tag{36}$$

When compared with Eq. (8), this equation implies that the function $g(x)$ must satisfy

$$\frac{dg(x)}{dx} + 2xg(x) = \sqrt{\frac{\pi}{2}}\, e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \tag{37}$$

which can be solved to find

$$g(x) = \frac{\pi}{4} e^{-x^2} \operatorname{erfi}^2\left(\frac{x}{\sqrt{2}}\right) \tag{38}$$

TRANSFORMATION APPROACH

In order to obtain a deeper understanding of the results derived by the power series method, we present another approach. We start by expressing the SHO (1) in terms of its Lagrangian

$$L_{SHO} = \frac{\dot{X}^2}{2} - \frac{X^2}{2} \tag{39}$$

Now, putting the requirement of the equivalence between the Lagrangian of the SHO (39) and the one corresponding to the oscillator under consideration (7), we conclude that the following should be satisfied

$$\dot{X} = e^{\frac{x^2}{2}} \dot{x}, \tag{40}$$

$$\frac{X^2}{2} = e^{x^2} g(x) \tag{41}$$

Equation (40) gives

$$dX = e^{\frac{x^2}{2}} dx \tag{42}$$

and its integration yields (see Eq. (34))

$$X = \sqrt{\frac{\pi}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \tag{43}$$

Based on Eq. (41), we find

$$g(x) = \frac{\pi}{4} e^{-x^2} \operatorname{erfi}^2\left(\frac{x}{\sqrt{2}}\right) \tag{44}$$

which agrees with the result (38) obtained by using the power series method.

As the final confirmation, let us use Eq. (40) and find its time derivative

$$\ddot{X} = \ddot{x} e^{\frac{x^2}{2}} + \dot{x}^2 x e^{\frac{x^2}{2}} \tag{45}$$

Substituting Eqs. (45) and (43) into the equation of motion of the SHO (1), we obtain

$$\ddot{x} + x\dot{x}^2 + \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) = 0 \tag{46}$$

which is in agreement with the expression for the straight-line backbone differential equation (36) obtained by using the power series method.

NUMERICAL CONFIRMATION

To provide numerical confirmations of the results obtained analytically, the equation of motion (36), i.e. (46), was solved numerically for different initial conditions and plotted in Fig. 2 (a solid line is obtained for $x(0)$=1, a dashed line for $x(0)$=0.5 and a dotted line for $x(0)$=0.1, while in all cases $\dot{x}(0)$=0). The time histories presented in Fig. 2 confirm that the period of vibration is independent of the amplitude and it is equal to 2π. Thus, the oscillator modelled by the Lagrangian (7) with $g(x)$ being defined by Eq. (38), with the equation of motion given by Eq. (36),

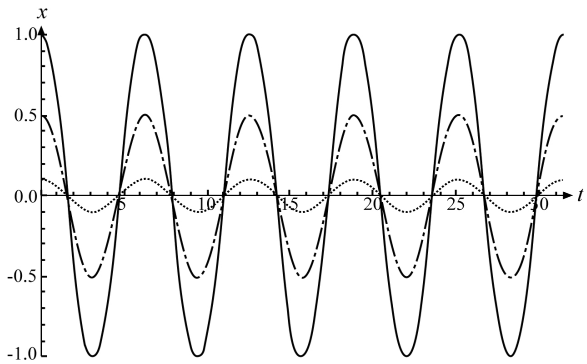

FIGURE 2. Numerically obtained time responses of the equation of motion (36), i.e. (46), for different initial conditions, $\dot{x}(0)=0$ and: $x(0)=1$ (solid line), $x(0)=0.5$ (dashed line) and $x(0)=0.1$ (dotted line).

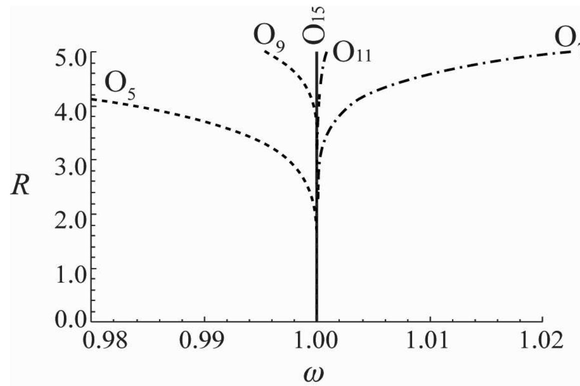

FIGURE 3. Numerically obtained backbone curves of the oscillator (31) with the coefficients defined by Eq. (30) for a different number of odd-powered terms (the highest power is indicated in the subscript).

possesses the property of isochronicity and has a straight-line backbone curve shown in Fig. 1.

An alternative way to achieving isochronicity by utilizing the imaginary error function in the equation of motion, is to use its power series representation (32). An important question is how many powered-form terms are needed for such an oscillator to exhibit a straight-line backbone curve. To answer it, the equation of motion (31) with the coefficients defined by Eq. (30) was solved numerically and the frequency ω was extracted from the time response for various values of the initial amplitude R. This was done for a different number of odd-powered polynomial terms in the sum. The backbone curves of the corresponding oscillators (O_j) are plotted in Fig. 3, where the subscript j denotes the highest power included into the sum. Fig. 3 shows that the oscillator O_5 has a softening backbone curve. With additional terms in the sum, the backbone curve gradually unbends, alternating its shape between the one corresponding to hardening and softening behaviour. The oscillator O_{15} and those with higher powers of nonlinearity exhibit a straight-line vertical backbone curve on the region of R considered.

CONCLUSIONS

This work has been motivated by the fact that the oscillators with a straght-line vertical backbone curve, when forced, do not exhibit hysteresis and jump phenomena. We have investigated the possibility of designing a differential equation corresponding to a conservative isochronous nonlinear oscillator, whose frequency would be amplitude-independent and its backbone curve would consequently be vertical and straight. To accomplish this aim, we have used two approaches: the power series approach based on Lindstedt's method and the transformation approach that establishes the equivalence between the Lagrangian of the simple harmonic oscillator, which is known to be isochronous, and a new oscillator. Both approaches have given the same results in terms of an exact expression for the straight-line backbone differential equation that contains the imaginary error function and the corresponding Lagrangian.

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