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FREQUENCY LOCKING IN A FORCED MATHIEU-VAN DER POL SYSTEM

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ABSTRACT

In this paper we investigate the dynamics of a Mathieu-van der Pol equation, which is forced both parametrically and nonparametrically. It is shown that the steady state response can consist of either 1:1 frequency locking, or 2:1 subharmonic locking, or quasiperiodic motion. The system displays hysteresis when the forcing frequency is slowly varied. We use perturbations to obtain a slow flow, which is then studied using the bifurcation software package AUTO. This study was motivated by an application to a MEMS device.

INTRODUCTION

This paper concerns the following differential equation, which may be thought of as a forced Mathieu-van der Pol equation:

$$\ddot{x} + (1 + \epsilon\alpha \ \cos 2\omega t)x - \epsilon\dot{x}(1 - x^2) = \epsilon F \ \cos\omega t \qquad (1)$$

where $\epsilon\alpha$ is the magnitude of parametric forcing applied at frequency 2ω, and ϵF is the magnitude of nonparametric forcing applied at frequency ω. ϵ is a small parameter which will be used in the perturbation method. Eq.(1) is a combination of two well known dynamical equations. The first results from a van der Pol (vdP) equation term (Nayfeh and Mook, 1979), $-\epsilon\dot{x}(1 - x^2)$, which, in the absence of forcing, leads to a steady state vibration called a limit cycle. For small values of ϵ the limit cycle has frequency close to 1, which is the frequency of the unforced linear oscillator. A study of entrainment in a forced VdP equation is given in (Rand, 2004) where it is shown that in the absence of parametric forcing ($\alpha = 0$), eq.(1) exhibits entrainment when $\omega \approx 1$. A second well known equation results from a Mathieu equation term (Nayfeh and Mook, 1979), $(1 + \epsilon\alpha\cos 2\omega t)x$, which, in the absence of the vdP term, renders the origin unstable when the parametric forcing frequency 2ω is close to the twice the frequency of the unforced linear oscillator. When the vdp term is added, the resulting limit cycle can be entrained at a 2:1 subharmonic. Thus in the case of both parametric and nonparametric forcing, eq.(1) can exhibit entrainment when $\omega \approx 1$.

Looking ahead, we may expect eq.(1) to exhibit the following two types of dynamical behavior:

- A quasiperiodic motion corresponding to the two frequencies of a) the limit cycle , and b) the forcing terms. This case is expected if the forcing amplitude is too small to produce entrainment.
- A periodic motion if the forcing amplitudes are sufficiently large. This could result from either 1:1 resonance if the nonparametric forcing frequency, ω, is close to 1, and/or 2:1 subharmonic resonance if the parametric forcing frequency, 2ω, is close to 2.

Our motivation for studying eq.(1) comes from previous studies of a MEMS device consisting of a thin, planar, radio frequency resonator (Zalalutdinov et al., 2003a),(Zalalutdinov et al., 2003b),(Pandey, 2005). These devices have been shown to self-oscillate in the absence of external forcing, when illuminated by a DC laser of sufficient amplitude. This system can also be forced externally either parametrically, by modulating the incident laser or nonparametrically, by using a piezo drive at the natural frequency of the device. In the presence of external forcing of sufficient strength and close enough in frequency to that of the unforced oscillation, the device will become frequency locked. The model presented in (Zalalutdinov et al., 2003a),(Zalalutdinov et al., 2003b) consisted of a third order system of ODE's. Our interest in eq.(1) comes from it being a simpler model which still involves all of the relevant phenomenon, namely limit cycles, parametric excitation and nonparametric excitation.

1 NUMERICAL INTEGRATION

We begin by numerically integrating eq.(1) and displaying the results in Fig.1, which shows the response amplitude as a function of forcing frequency ω for parameters $\epsilon = 0.1$, $\alpha = 1$ and $F = 0.3$. Quasiperiodic behavior (QP) is observed in the regions located approximately at $\omega < 0.97$ and $\omega > 1.03$. Periodic behavior at the forcing frequency is observed in the rest of the plot, corresponding to entrainment. As we sweep the frequency forward inside the entrained region, the amplitude jumps to a higher value at a frequency $\omega \approx 1.015$. No comparable jump is seen when the frequency is swept backward, indicating hysteresis.

2 PERTURBATION METHOD

In order to better understand the foregoing numerical results, and to study the effect of changing parameters, in this section we use the two variable expansion method (also known as the method of multiple scales) to obtain an approximate analytic solution. The idea of this method is to replace time t by two time scales, $\xi = \omega t$, called stretched time, and $\eta = \epsilon t$, called slow time. The forcing frequency ω is expanded around the natural frequency of the oscillator $(= 1)$, i.e.

$$\omega = 1 + k_1\epsilon + O(\epsilon^2) \tag{2}$$

where k_1 is a detuning parameter at order ϵ. Next, x is expanded in a power series in ϵ:

$$x = x_0(\xi, \eta) + \epsilon x_1(\xi, \eta) + O(\epsilon^2) \tag{3}$$

Substituting (2),(3) into (1) and collecting terms gives:

$$x_{0\xi\xi} + x_0 = 0 \tag{4}$$

$$x_{1\xi\xi}+x_1 = -2k_1x_{0\xi\xi}+(1-x_0^2)x_{0\xi}-\alpha x_0 \cos 2\xi+F\cos\xi \tag{5}$$

We take the solution to eq.(4) in the form:

$$x_0(\xi, \eta) = A(\eta)\ \cos\xi + B(\eta)\ \sin\xi \tag{6}$$

Substitution of (6) into (5) and removal of secular terms gives the following slow flow:

$$A' = -k_1B + \frac{A}{2} - \frac{A}{8}(A^2+B^2) - \frac{B}{4}\alpha \tag{7}$$

$$B' = k_1A + \frac{B}{2} - \frac{B}{8}(A^2+B^2) - \frac{A}{4}\alpha + \frac{F}{2} \tag{8}$$

Equilibrium points in the slow flow (7),(8) correspond to periodic motions in eq.(1), whereas limit cycles in the slow flow correspond to quasiperiodic motions in (1).

3 INVARIANCES OF THE SLOW FLOW

The slow flow (7),(8) contains 3 parameters: detuning k_1, parametric forcing amplitude α, and nonparametric forcing amplitude F. We shall be interested in understanding how the phase portrait of the slow flow is determined by these parameters. However, before discussing this we note that eqs.(7),(8) exhibit some invariances which permit useful conclusions to be drawn. For example, eqs.(7),(8) remain unchanged when A, B and F are replaced respectively by $-A$, $-B$ and $-F$. Since such a change does not alter the nature of the phase portrait, we see that *we may consider* $F \geq 0$ *without loss of generality.*

In addition, we see that eqs.(7),(8) remain unchanged when A, k_1 and α are replaced respectively by $-A$, $-k_1$ and $-\alpha$. Since changing the sign of A does not alter the nature of the phase portrait, we see that *we may consider* $\alpha \geq 0$ *without loss of generality* since negative α corresponds to flipping a $k_1 - F$ bifurcation diagram (such as shown in Fig.2) about a vertical axis (so that k_1 is replaced by $-k_1$).

4 AUTO BIFURCATION SOFTWARE

In order to understand how the dynamical behavior varies as k_1, α, and F are changed, we used the AUTO bifurcation software package (Doedel et al., 2002).

Fig.2 shows the results of the AUTO analysis for $\alpha = 1$, where k_1 and F are varied. Region A contains 5 slow flow equilibria, consisting of 2 sinks, 2 saddles and 1 source, i.e., only 2 are stable. These stable equilibria correspond to frequency locked periodic motions in eq.(1). The presence of two such steady states signals the possibility of hysteresis.

This same (stable) steady state occurs in region D, which lies above region A in Fig.2. The difference between these two regions is that D contains only 3 slow flow equilibria, namely 2 sinks and a saddle. As we cross the curve separating A and D, two of the saddles in A merge with the source in A in a pitchfork bifurcation, leaving a single saddle in their place.

We next consider regions E and B which lie to the left and right of region A in Fig.2. The slow flow phase portrait for points in these two regions are qualitatively the same, consisting of 3 slow flow equilibria, namely a source, a saddle and a sink. Only one of these slow flow equilibria is stable, and corresponds to a periodic motion at the forcing frequency in eq.(1). This same (stable) steady state occurs in region C_2, which lies above region D in Fig.2. The difference between region C_2 and regions E and B is that C_2 contains just 1 slow flow equilibrium point, namely a sink. As we cross the curve separating C_2 from one of the regions E, D or B below it (each of which contains 3 slow flow equilibria), a saddle-node bifurcation occurs leaving a single sink in region C_2.

We have now discussed all regions in Fig.2 except for regions C_1 which lie in the lower left and right portions of Fig.2. Regions C_1 contain a single unstable slow flow equilibrium point, namely a source. However, unlike the other regions in Fig.2, regions C_1 also contain a stable slow flow limit cycle. This motion corresponds to a stable quasiperiodic motion in eq.(1). Hopf bifurcations occur along the curves separating regions C_1 and C_2.

We offer the following summary of predicted (stable) steady state behavior of eq.(1): In regions A and D we have 2 distinct stable periodic motions; in regions E, C_2 and B we have a single stable periodic motion; and in regions C_1 we have a stable quasiperiodic motion.

The discussion thus far has fixed α at unity (Fig.2). We next look at the effect of changing α. See Fig.3. We see that the width of the region corresponding to 2 distinct steady state periodic motions (regions A and D) decreases as we decrease α. In addition the regions B and E become more symmetrical. For $\alpha = 0.2$ we see that the region D has disappeared and the regions E and B have merged to give just one region. At $\alpha = 0$ the region with 2 stable steady states has disappeared. This case corresponds to nonparametric periodic forcing of a van der Pol oscillator, and has been discussed in (Rand, 2004).

Thus the presence of the regions A and D which contain 2 stable periodic motions may be associated with the parameter α. Since α is the coefficient of the parametric excitation term which has frequency 2ω, we may associate these regions with 2:1 subharmonic response. This is in contrast to regions E, B and C_2, which may be associated with 1:1 frequency locking.

5 CONCLUSION

We have shown that the steady state response of the forced Mathieu-van der Pol equation (1) can consist of either 2:1 subharmonic periodic motion or 1:1 periodic motion or quasiperiodic motion, depending on the forcing frequency and the forcing amplitudes, both parametric and nonparametric.

The hysteresis observed in numerical simulations with slowly varying forcing frequency (Fig.1) is explained by changes in the nature of the steady state due to bifurcations in the slow flow equilibria. For example, the jump upwards in Fig.1 is due to passage from region A to region B in the process of which a saddle-node bifurcation occurs and a stable periodic motion disappears.

Our findings may be summarized briefly in words (for fixed α) by stating that parametric excitation is most important when ω is close to the unforced frequency of the oscillator (here taken as unity), or, in other words, when the detuning k_1 is close to zero, and when F is small. Nonparametric forcing takes over when $|k_1|$ is a little larger or when F is larger. Quasiperiodic behavior occurs when $|k_1|$ becomes sufficiently large for given F.

6 ACKNOWLEDGMENT

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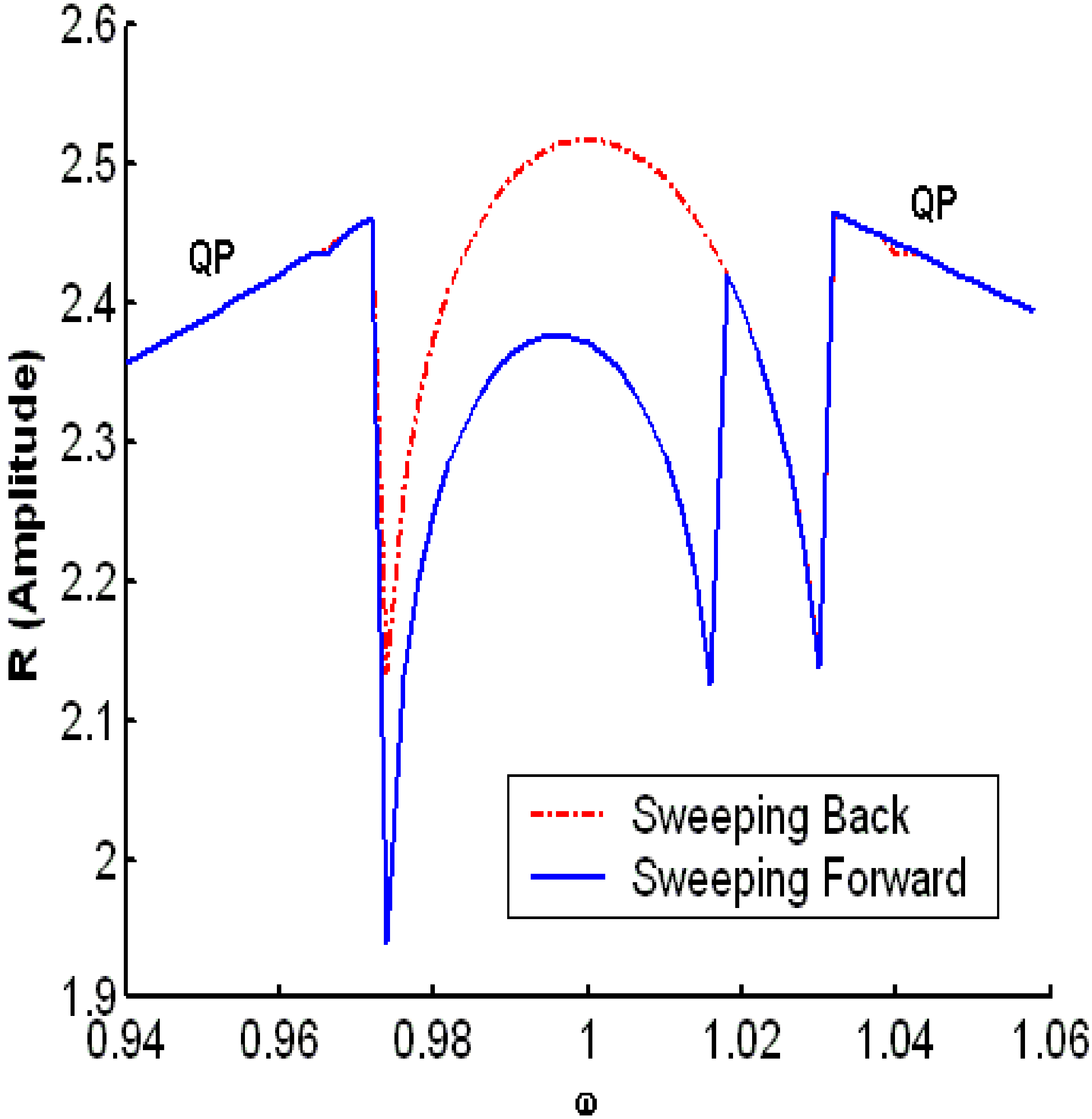

Figure 1: Results of numerical integration of eq.(1) for parameters $\epsilon = 1$, $\alpha = 1$ and $F = 0.3$. Response amplitude R is plotted against forcing frequency ω. Quasiperiodic behavior (QP) is observed in the regions located approximately at $\omega < 0.97$ and $\omega > 1.03$. Periodic behavior at the forcing frequency is observed in the rest of the plot, with hysteresis as shown.

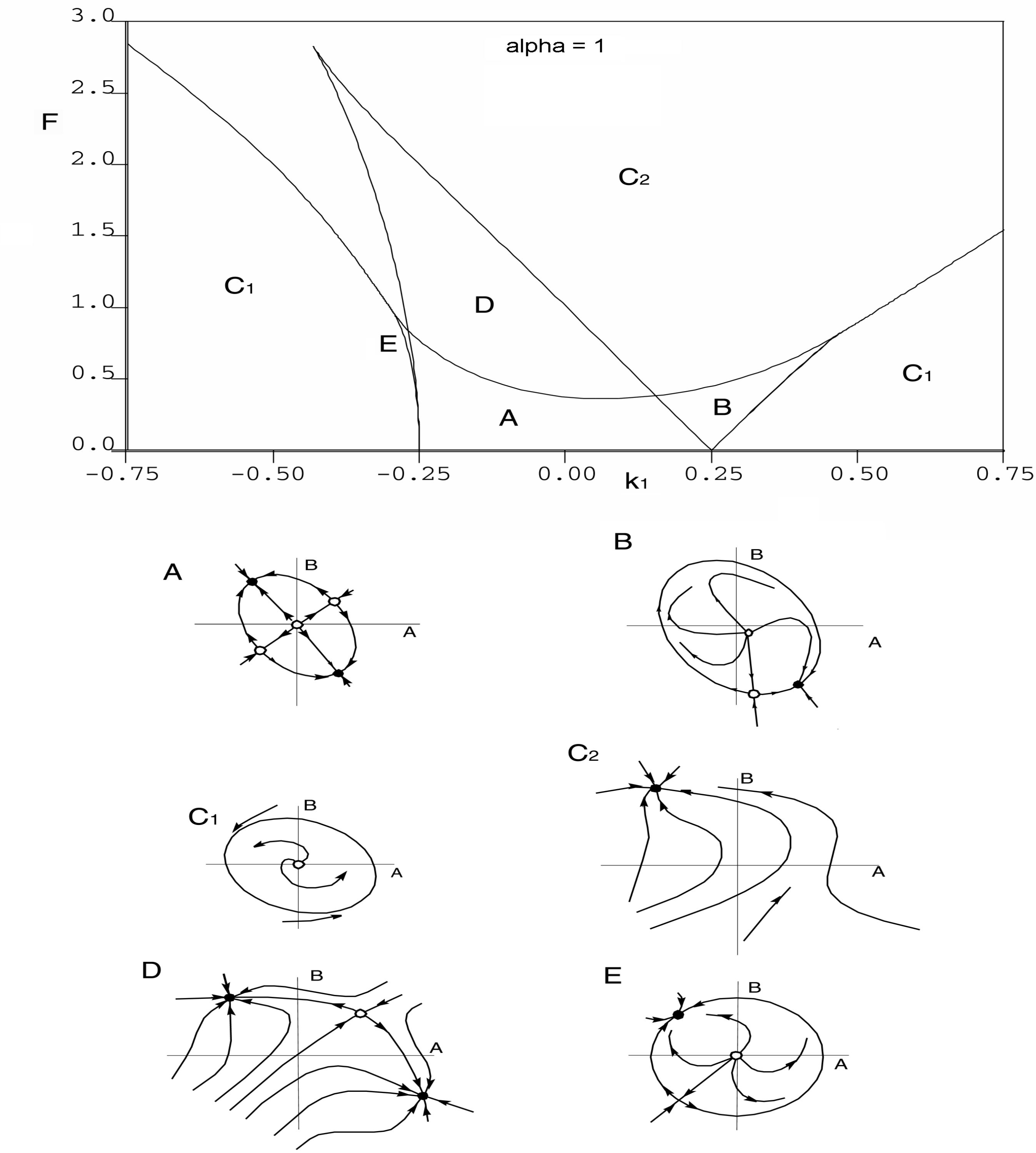

Figure 2: Behavior of the slow flow (7),(8). Bifurcation curves (obtained using AUTO) and sketches of corresponding slow flow phase portraits are displayed for $\alpha = 1$.

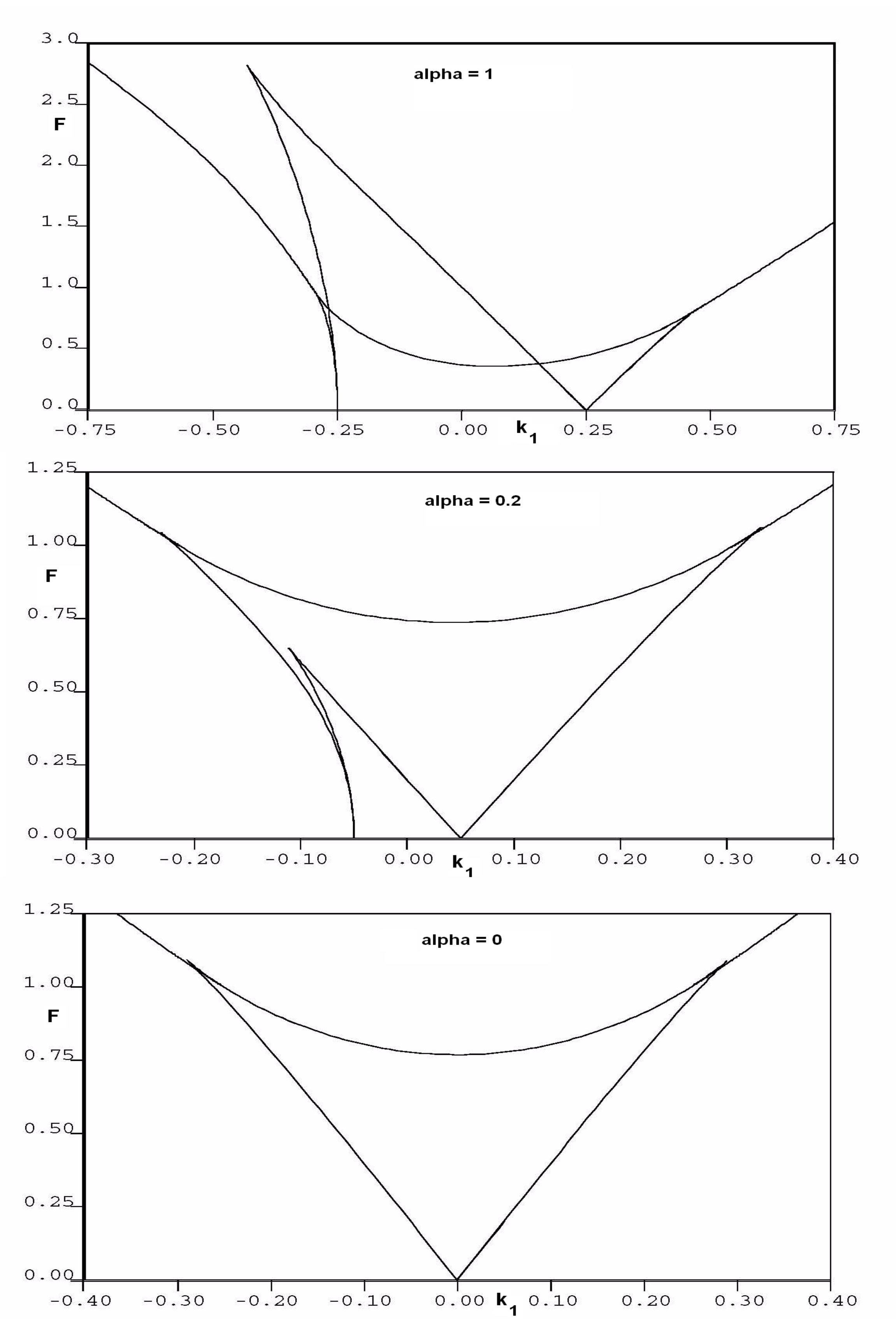

Figure 3: Bifurcation curves (obtained using AUTO) for slow flow (7),(8), for $\alpha = 1, 0.2$ and 0.