

Sequences and Recursion

Some techniques to consider:

Continuity to Evaluate the Possible Value of a Limit: Continuous functions preserve convergent sequences, so if the limit exists, it's value can often be determined by taking the limit in a defining relation. Existence of the limit must still be established.

Bounded Monotonic Sequences: Such sequences of real numbers automatically converge. And bounded sequences have convergent subsequences.

Contraction Mapping Theroem: An iteration $x_{n+1} = f(x_n)$ where $f : [a, b] \rightarrow [a, b]$ with $|f'(x)| < 1$ must converge to a unique fixed point. A common proof technique (as for the CMT) to establish $\lim_{n \rightarrow \infty} a_n = L$ is to compare $|a_{n+1} - L|$ with $|a_n - L|$.

Linear Recursion Relations: If constant coefficient homogeneous, solutions can be translated into computing powers of a square matrix, and these can be computed by diagonalization. (*Seeking solutions $a_n = c\lambda^n$ will identify the eigenvalues without writing down the matrices.*) As with differential equations, the method of undetermined coefficients can be used in the inhomogeneous case.

Manipulate a Recursive Relation into Something Simpler: Suggestions for this may be driven either by the form of the recursive relationship, or by what you are asked to establish.

Mathematical Induction: Often experimentation with the first few terms helps here. Also remember the "strong form" in which for the inductive step, you assume *all* previous cases.

Obtain a Recursive Relationship: If this is not already present in the presentation of the sequence.

Techniques as with Summation: One studies an infinite series $\sum_{n=0}^{\infty} a_n$ by looking at the limit of the partial sums S_n (and $a_n = \sum_{k=0}^n S_n - S_{n-1}$) so infinite sum and sequence limit problems can be interchanged. Thus it is not surprising that techniques like Taylor approximation and generating functions also appear here.

Number Theory Facts: These come up because number theory often gives rise to interesting sequences. Facts such as the Chinese Remainder Theorem (letting you reduce many modular computations to prime power moduli) and the application of Euler's Totient function to computing powers show up a fair bit.

Identify a Property Which is Preserved in the Sequence: For example the nonzeroness or parity of the terms in the case of an integer sequence.

Combinatorial Enumeration: Just as problems which are essentially number theory show up here, many combinatorics problems are fundamentally about properties of sequences.

The problems appearing in the October 13 handout were 85A3, 85A4, 87A2, 90A2, 91B1, 92B3, 92B5, 93A2, 93A6, 97A6, 97B5, 98B5, and 99A6.

If you'd like to seek some other problems of this sort, possibilities you can find online include 93A6, 65B4, 66A3, 66A4, 68B6, 69A2, 69A6, 69B3, 70A4, 79A3, 80B3, 80B6, 81A1, 82B5, 84B6, 46B4, 46B5, 47A1, 47A5, 48B3, 49A5, 50A3, 50B5, 52B7, 53A6, 53B2, 55A1, 56A1, 58A1, 58A2, 58B1, 61B7, 63B5, 64A4, and 64B1.

A good online reference to problems and solutions is

<http://www.kalva.demon.co.uk/putnam.html>

by John Scholes which has almost all (!) problems and solutions.