PROBLEM SET #1

(1) There is an infinite row of lightbulbs labeled 0, 1, 2, ... Every lightbulb has a button to turn it on or off, and starts in the off position. A person presses the buttons on the even numbers; then presses the buttons on the multiples of 3; then the multiples of 4 and so on. At the end of this process, which lightbulbs remain on?

(2) Let $I_1, I_2, \ldots, I_n$ be $n$ intervals on the real line, such that $I_r \cap I_s \neq \emptyset$ for all $r, s \in \{1, \ldots, n\}$. Prove that ALL intervals have a point in common.

(3) Let $C_1, C_2$ be two disjoint circles in the plane. From the center of $C_1$ draw the tangent lines to $C_2$ and call $P_1, Q_1$ their intersections with $C_1$. In a similar manner, construct $P_2, Q_2$ on $C_2$. Show that $P_1Q_1 = P_2Q_2$.

(4) Consider a set $S$ and a binary operation $*$ such that if $a, b \in S$ then $a*b \in S$. Suppose that $(a*b)*a = b$ for all $a, b \in S$. Prove that $a*(b*a) = b$ for all $a, b \in S$.

(5) Let $m, n$ be two positive integers. Prove that gcd$(m, n)$ divides lcm$(m, n)$.

(6) (a) Each square of a $4 \times 7$ rectangle is colored black or white. Prove that for any such coloring, there is a rectangle with the same color on all 4 corner squares.

(b) Show a coloring of a $4 \times 6$ rectangle such that every sub-rectangle has corner squares of both colors.

(7) What is the largest real number $x$ such that $a^{a^{\ldots}}$ converges to a finite real number?

(8) Given a positive integer $n$, let $p(n)$ be the product of the non-zero digits of $n$. Compute $S = p(1) + p(2) + \ldots + p(10^{10})$.

(9) Let $A$ be a curve that winds through the interior of a sphere $S$ of radius 1. Suppose that the endpoints of $A$ are on the boundary of $S$ and that the length of $A$ is less than 2. Show that there is a hemisphere of $S$ that does not intersect $A$.

(10) Prove that at any party there must be two people that have the same number of friends present (friendship is assumed to be mutual).