# Solution Hints v2

1. Show that if $0 < a \leq c$ and $0 < b \leq d$ then
$$ad + bc \leq ab + cd$$
and use this to show that $\Sigma_{n=1}^{N} x_n y_n$ can be increased by rearrangement if it is not already of the form $\Sigma_{n=1}^{N} x_n^* y_n^*$.

2. Reduce the equation modulo $a$ and modulo $b$. Take the resultant description of all potential integer solutions as
$$\begin{aligned} m &= kb - 1 \\ n &= la - 1 \end{aligned}$$
and plug these into the original equation quickly seeing that $m$ and $n$ both positive is not possible.

3. *There was a typo on the exam as printed in forgetting to include the condition $\epsilon < 1$.*
The mean value theorem readily shows that
$$b_n = (n+1)^{1-\epsilon} - n^{1-\epsilon}$$
approaches 0 as $n \to \infty$. It also shows that the sequence $(b_k)_{k=1}^{\infty}$ is a decreasing sequence as $n$ increases. So given any desired accuracy $\delta$, to well approximate $x$, consider the telescoping series
$$S_M = \Sigma_{k=n}^{M} b_k = (M+1)^{1-\epsilon} - n^{1-\epsilon}$$
starting with an $n$ so that $b_n < \min(\delta, x)$. Here we have introduced $S_M$ to denote the sum. For any $M \geq n$, the telescoping series $S_M$ gives a number of the requested form. As $M \to \infty$, the sum $S_M$ approaches $\infty$, (this is where $\epsilon < 1$. is relevant), so eventually surpasses $x$. Letting $M$ be the first index for which

$S_M$ at least $x$, we see that this $S_M$ is the requested approximation.

4. The possibilities $k = 1$ and $k = 2$ are readily ruled out. The example $x(x-1)(x+1)(x-2)(x+2)$ shows $k = 3$ suffices.

5. If $a \geq b$, the enclosing square S must have a side length $x$ of at least $2a$ just to contain the circle of radius $a$.

   To get an additional condition, note that the center $O_A$ of circle $A$ of radius $a$ must be inside the subsquare $S_A$ formed from square $S$ by moving a distance $a$ inward along each edge. (Otherwise circle $A$ would in part be outside $S$.)

   Similarly center $O_B$ gives rise to a subsquare $S_B$.

   Now if one draws a little picture and thinks about the diagonal of square $S$ (which includes the diagonals of $S_A$ and $S_B$ as well), the condition of $O_A$ being in $S_A$ and $O_B$ being in $S_B$ quickly leads to the distance between $O_A$ and $O_B$ being bounded by the distance from one corner of $S_A$ to the opposite corner of $S_B$.

   Computing this distance (which is along the diagonal mentioned earlier) leads to $\sqrt{2}(x-a-b)$. On the other hand the distance from $O_A$ to $O_B$ has to be at least $a+b$, so we have the additional inequality

   $$\sqrt{2}(x-a-b) \geq a+b$$

   or

   $$x \geq (a+b)\left(1+\frac{1}{\sqrt{2}}\right).$$

   It's now clear that the minimum sidelength is

   $$x = \min\left(2a, (a+b)\left(1+\frac{1}{\sqrt{2}}\right)\right)$$

   making sure that both square $S_A$ is nonempty (so the circle centered at a corner of $S_A$ fits) and that there is enough room along the diagonal to place the center of the circle of radius $b$.