Solution Hints v2

1. Show that if $0 < a \leq c$ and $0 < b \leq d$ then
   \[ ad + bc \leq ab + cd \]
   and use this to show that $\sum_{n=1}^{N} x_n y_n$ can be increased by rearrangement if it is not already of the form $\sum_{n=1}^{N} x'_n y'_n$.

2. Reduce the equation modulo $a$ and modulo $b$. Take the resultant description of all potential integer solutions as
   \[ m = kb - 1 \]
   \[ n = la - 1 \]
   and plug these into the original equation quickly seeing that $m$ and $n$ both positive is not possible.

3. There was a typo on the exam as printed in forgetting to include the condition $\epsilon < 1$.
   The mean value theorem readily shows that $b_n = (n + 1)^{1-\epsilon} - n^{1-\epsilon}$ approaches 0 as $n \to \infty$. It also shows that the sequence $(b_k)_{k=1}^{\infty}$ is a decreasing sequence as $n$ increases. So given any desired accuracy $\delta$, to well approximate $x$, consider the telescoping series
   \[ S_M = \sum_{k=n}^{M} b_k = (M + 1)^{1-\epsilon} - n^{1-\epsilon} \]
   starting with an $n$ so that $b_n < \min(\delta, x)$. Here we have introduced $S_M$ to denote the sum. For any $M \geq n$, the telescoping series $S_M$ gives a number of the requested form. As $M \to \infty$, the sum $S_M$ approaches $\infty$, (this is where $\epsilon < 1.$ is relevant), so eventually surpasses $x$. Letting $M$ be the first index for which
$S_M$ at least $x$, we see that this $S_M$ is the requested approximation.

4. The possibilities $k = 1$ and $k = 2$ are readily ruled out. The example $x(x - 1)(x + 1)(x - 2)(x + 2)$ shows $k = 3$ suffices.

5. If $a \geq b$, the enclosing square $S$ must have a side length $x$ of at least $2a$ just to contain the circle of radius $a$.

To get an additional condition, note that the center $O_A$ of circle $A$ of radius $a$ must be inside the subsquare $S_A$ formed from square $S$ by moving a distance $a$ inward along each edge. (Otherwise circle $A$ would in part be outside $S$.)

Similarly center $O_B$ gives rise to a subsquare $S_B$.

Now if one draws a little picture and thinks about the diagonal of square $S$ (which includes the diagonals of $S_A$ and $S_B$ as well), the condition of $O_A$ being in $S_A$ and $O_B$ being in $S_B$ quickly leads to the distance between $O_A$ and $O_B$ being bounded by the distance from one corner of $S_A$ to the opposite corner of $S_B$.

Computing this distance (which is along the diagonal mentioned earlier) leads to $\sqrt{2}(x - a - b)$. On the other hand the distance from $O_A$ to $O_B$ has to be at least $a + b$, so we have the additional inequality

$$\sqrt{2}(x - a - b) \geq a + b$$

or

$$x \geq (a + b) \left(1 + \frac{1}{\sqrt{2}}\right).$$

It’s now clear that the minimum sidelength is

$$x = \min\left(2a, (a + b) \left(1 + \frac{1}{\sqrt{2}}\right)\right)$$

making sure that both square $S_A$ is nonempty (so the circle centered at a corner of $S_A$ fits) and that there is enough room along the diagonal to place the center of the circle of radius $b$. 2