Putnam Practice Exam v1

4:45pm – 6:15pm, October 1st, 2008

1. Let $(x_n)_{n=1}^N$ and $(y_n)_{n=1}^N$ be two finite length sequences of positive real numbers. Denote by $(x_n^*)_{n=1}^N$ and $(y_n^*)_{n=1}^N$ the same sequences, but written in a decreasing order. Show that
$$\Sigma_{n=1}^N x_n y_n \leq \Sigma_{n=1}^N x_n^* y_n^*.$$

2. Let a and b be positive integers whose greatest common divisor $\gcd(a, b)$ is 1. Show that the equation
$$am + bn = ab - a - b$$
never has positive integer solutions m and n.

3. Let $1 > \epsilon > 0$ be fixed. Show that every real number can be approximated arbitrarily well by differences of the form
$$n^{1-\epsilon} - m^{1-\epsilon}$$
for some positve integers m and n.
By arbitrarily good approximation of x, we mean that given any $\delta > 0$, there are positive integers m and n so that
$$|x - \left(n^{1-\epsilon} - m^{1-\epsilon}\right)| < \delta.$$

4. Let k be the smallest positive integer with the following property:

   There are distinct integers m_1, m_2, m_3, m_4, and m_5 such that the polynomial
   $$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$
   has exactly k nonzero coefficients.

   Find, with proof, a set of integers m_1, m_2, m_3, m_4, and m_5 for which this minimum is achieved.

5. For any two positive numbers a and b, find the side length of the smallest square containing two non-overlapping circles of respective radii a and b. (By non-overlapping, we mean that the interiors of the two circles do not intersect.)