

# Polynomials

Among the problems today:

**Easiest:** 85B1, 86A1, 86B2, 87A1, 99A1, and 99A2.

**Moderate:** 85B2, 86B5, 87A4, 87B3, 88B2, 88B6, 89A3, 91A3, 92B4, 94B2, 99A5 and 99B2.

**Hardest:** 85A6, 86A6, 86B3, and 90 B5.

*(Revised Version - Details of the Newton formulae derivation are at the end of this writeup.)*

Some techniques to consider:

**Division Algorithm:** Given nonzero one variable polynomials  $p$  and  $q$ , there are unique polynomials  $f$  and  $r$  so that  $p = qf + r$  with degree  $r$  less than degree of  $q$ .

**Roots and Factorization:** A one variable polynomial is

$$p(x) = c \prod (x - r_i)$$

where the  $r_i$  are the roots. This also says that the coefficients of a monic polynomial are up to sign the elementary symmetric functions  $\sigma_i$  of the roots; i.e.  $p(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} \dots + (-1)^n \sigma_n$ . (Another form of generating function for the  $\sigma_i$  would be  $\prod_{i=1}^n (1 + tr_i) = \sum_{i=0}^n \sigma_i t^i$ .)

**Polynomial Inequalities:** in one variable readily solvable via the above factorization.

**Logarithmic Derivative**  $\frac{p'(x)}{p(x)} = \sum \frac{1}{x-r_i}$  can be used to infer some properties of  $p'$  from those of  $p$ . Multiplying this equation by  $p$ , bringing in the symmetric functions of the roots, and expanding  $\frac{1}{x-r_i}$  via geometric series gives the Newton Formulae relating the elementary symmetric functions to the easier to compute symmetric sums  $s_k = \sum r_i^k$ .

**Intermediate Value Theorem:** can be used to argue for the existence of roots.

**Complex Polynomials:** can encode more complicated real systems. For example

$$\begin{aligned}x^2 - y^2 &= 1 \\2xy &= 1\end{aligned}$$

can be viewed as the real and imaginary parts of  $z^2 = 1+i$  where  $z = x+iy$ . Similarly

$$\frac{x}{x^2 + y^2} = 1$$

$$\frac{y}{x^2 + y^2} = 1$$

would correspond to  $\frac{1}{z} = 1 - i$ . Similarly trigonometric formulae arise. For example the polar representation  $z = re^{i\theta} = r(\cos \theta + i \sin \theta)$  means that  $2 \cos \theta = z + 1/z$  and  $2 \sin \theta = z - 1/z$  when  $z$  lies on the unit circle  $|z| = 1$ .

**Polynomial Relations:** can come from the principle that a homogeneous linear system of equations with a nontrivial solution has a zero determinant.

**Multiplicative Homomorphisms:** such as evaluation at a point, reduction mod  $p$ , or replacing  $x$  by  $\frac{1}{x}$  in a polynomial can be applied to polynomial relationships, and the results inferred from the behavior on simple factors.

**Specialization:** to “simpler” places can be used to infer the consequences of functional equations involving polynomials.

**Differentiation :** of polynomial relationships can be used to recover coefficients (Taylor approximation idea), or to establish results such as Euler’s formula  $nP(x) = \sum x_i \frac{\partial P}{\partial x_i}$  for a homogeneous polynomial of degree  $n$ .

**Diophantine Equations:** can sometimes be solved by describing solutions in terms of geometrically natural parameters (such as the slope of a well chosen line through a solution.) For example pythagorean triplets arise this way.

**Absolute Value:** can be readily used to express max and min; namely  $\max(a, b) = \frac{a+b+|a-b|}{2}$ .

**Experimentation:** with smaller cases sometimes points the way for inductive arguments that are easy to establish once “guessed.”

A good online reference to problems and solutions is

<http://www.kalva.demon.co.uk/putnam.html>

by John Scholes which has almost all (!) problems and solutions.

## Details of the Newton Formulae

Start with the generating function

$$P(t) = \prod_{i=1}^n (1 + tr_i) = \sum_{i=0}^n \sigma_i t^i.$$

Then, for example by taking the logarithm of the left hand equality and differentiating with respect to  $t$ ,

$$\frac{P'(t)}{P(t)} = \sum_{i=1}^n \frac{r_i}{1 + tr_i}.$$

Expanding the right hand side about  $t = 0$  (essentially a geometric series) leads to

$$\frac{P'(t)}{P(t)} = \sum_{j=0}^{\infty} (-1)^j t^j s_{j+1}$$

where  $s_j = \sum_{i=1}^n r_i^j$ . Clear denominator by multiplying by  $P(t)$ . One obtains

$$P'(t) = P(t) \sum_{j=0}^{\infty} (-1)^j t^j s_{j+1}.$$

Comparing coefficients of  $t^j$  for  $j \geq 0$  leads to the infinite triangular system of equations

$$\begin{aligned} \sigma_1 &= s_1 \\ 2\sigma_2 &= -s_2 + s_1\sigma_1 \\ 3\sigma_3 &= s_3 - s_2\sigma_1 + s_1\sigma_2 \\ \dots &= \dots \\ (j+1)\sigma_{j+1} &= (-1)^j s_{j+1} - s_j\sigma_1 + \dots + s_1\sigma_j \\ \dots &= \dots \end{aligned}$$

which can be recursively solved for  $\sigma_{j+1}$  leading to the expression of the elementary symmetric polynomials in terms of the symmetric sums  $s_i$ ; e.g.

$$\begin{aligned} \sigma_1 &= s_1 \\ \sigma_2 &= \frac{1}{2} (-s_2 + s_1^2) \\ \sigma_3 &= \frac{1}{3} \left( s_3 - s_2s_1 + s_1 \frac{1}{2} (-s_2 + s_1^2) \right) = \frac{1}{6} (2s_3 - 3s_1s_2 + s_1^3) \\ \dots &= \dots \\ \sigma_{j+1} &= \dots \end{aligned}$$