Induction

Induction is the tool we use when we want to prove that some property is valid for ALL numbers. It is a mistake to try to prove that 1 has the property, and 2 has the property, and 3 has the . . . we would never finish. Instead we prove 2 things:

a) The number 1 has the property.
b) IF the number $k$ has the property, then the number $k + 1$ has the property.

Let’s see how point b) reads when substituting $k$ with 1: “If 1 has the property, then $1 + 1 = 2$ has the property.” Since we proved that 1 does have the property, point b) shows that 2 also has the property.

Now we can substitute $k$ with 2 in point b) ”If 2 has the property, then $2 + 1 = 3$ has the property.” Since we just proved a moment before that 2 indeed has the property, it follows that 3 will too. This process repeats itself automatically.

As soon as we prove a) and b), we have a machine that shows (one step at a time) that every number has the desired property.

The best way to understand induction is with an example. We will prove a basic fact that is useful all on its own.

For any natural number $n$, the following equality holds:

\[ 1 + 2 + \ldots + n = \frac{n \cdot (n + 1)}{2}. \]  

First, let us verify that this is true in some cases.

\[ 1 = \frac{1 \cdot 2}{2}, \quad 1 + 2 = 3 = \frac{2 \cdot 3}{2}, \quad 1 + 2 + 3 = 6 = \frac{3 \cdot 4}{2}. \]

This helps us to see any patterns that may appear. We also established point a); the number 1 does satisfy property ($\star$). Now we prove point b).

SUPPOSE that ($\star$) is true for $k$. Then

\[ 1 + 2 + \ldots + (k + 1) = (1 + 2 + \ldots + k) + (k + 1) = \frac{k \cdot (k + 1)}{2} + (k + 1) = (k + 1) \left( \frac{k}{2} + 1 \right) = (k + 1) \left( \frac{k + 2}{2} \right) = \frac{(k + 1)(k + 2)}{2}. \]

That is, ($\star$) is also valid for $k + 1$.

Some things to remember:

- You must prove both a) and b) for the induction to be complete.
- You may need to start at a number $r$ other than 1. Then the induction proves that the property is valid for all numbers greater than $r$.
- Sometimes you will need strong induction. This means that instead of point b) you prove (see problem 5 for an example)
  b’) If all numbers from 1 to $k$ have the property, then $k + 1$ has the property.

Problems —→
1. Prove that \(1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2\) for all \(n\).

2. The Fibonacci numbers are defined by the recurrence \(F_1 = 1, F_2 = 1\) and \(F_{k+2} = F_{k+1} + F_k\) for \(k \geq 1\). Show that for every \(n \geq 1\) they satisfy
\[
F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.
\]

3. Show that all numbers in the sequence 1007, 10017, 100117, 1001117, 10011117, \ldots are divisible by 53.

4. Prove that for \(n \geq 6\), a square can be dissected into \(n\) smaller squares, not necessarily all of the same size.

5. Show that every positive integer can be written as the sum of distinct Fibonacci numbers.