Sample Freshmen Prize Exam Questions
(More Than Two Exam's Worth)

Problem 1. A box contains 7 white balls and 8 black balls. If 3 balls are drawn from the box at random, what is the probability of drawing 2 white balls and 1 black ball?

Problem 2. The lines normal to the parabola $y = x^2$ through the points (x, x^2), $x \neq 0$, intersect the y-axis at a set of points Y. What is the largest value of y that is NOT in the set Y.

Problem 3. Put $c_1 = 1$ and define $c_{n+1} = c_n + \frac{1}{c_n}$. Find $lim_{n \to \infty} c_n$.

Problem 4. How many 0's does 2000! end in?

Problem 5. Find the sum of all positive integral divisors of 9600.

Problem 6. A rectangle with sides parallel to the axes is drawn in the first quadrant region bounded by the x-axis and the curve $y = \frac{x}{x^2+1}$. The rectangle is then rotated about the x-axis to form a solid. What is the maximum possible volume of this solid?

Problem 7. The UN invited $2n$ diplomats for a dinner. Each of them has at most $n - 1$ enemies. Show that you can seat them at a round table so that nobody sits next to an enemy. (We assume that being an enemy is symmetric: if A is an enemy of B, then B is an enemy of A.)

Problem 8. Every point in the plane is colored either blue, red, or green. Show that there is a rectangle all of whose corners have the same color.

Problem 9. The number 31372668556835970837 is the 11^{th} power of some number n. Find n.

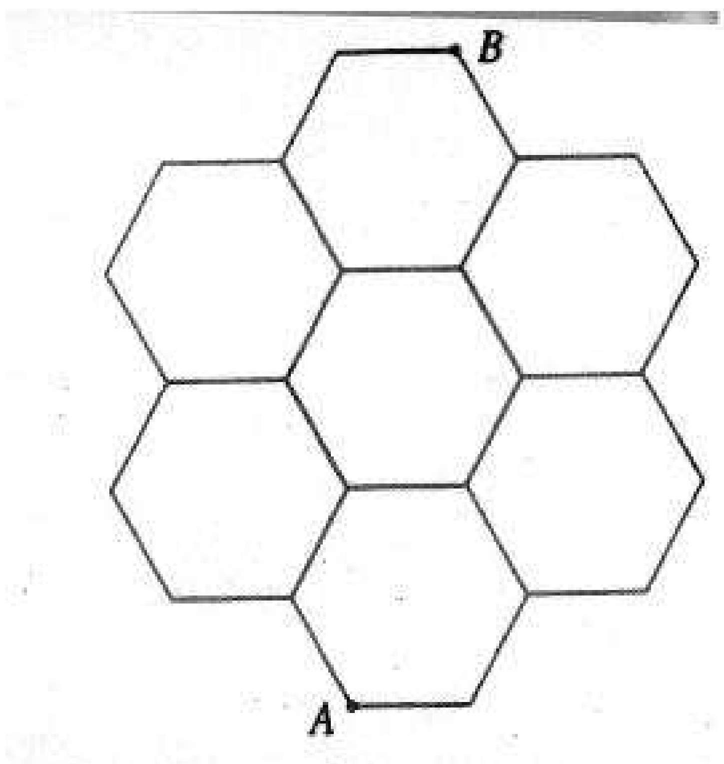
Problem 10. Starting with any real number x_0, a sequence of numbers $\{x_n\}_{n=0}^{\infty}$ is defined by $x_{n+1} = \cos(x_n)$ where x_n is measured in radians. Show that

$$\lim_{n\to\infty} x_n$$

exists and is independent of the initial value x_0.

Problem 11. The sum of the positive integers $\{a_1, \dots, a_r\}$ is $a_1 + \dots + a_r = 2005$. What is the largest possible product $a_1 \cdot \ldots \cdot a_r$ that can be formed under this condition?

Problem 12. A path from A to B in the figure is *valid* if it does not cross itself, and never moves downwards. How many valid paths are there?



Problem 13. Prove the inequality

$$\left(\frac{a}{b}\right)^b \leq \left(\frac{a+1}{b+1}\right)^{b+1} \qquad \text{for } a > 0,\ b > 0.$$

Problem 14. Show that

$$lim_{R\to\infty} \int_0^R \sin\left(t^2\right)\, dt$$

exists and is finite.

Problem 15. Let p be a prime number.

(a) Show that p divides $2^p - 2$.

(b) Show that p^2 divides $2^{p^2} - 2^p$.

Problem 16. Given are n points in space: $A_0, A_1, \ldots A_n = A_0$. Pick a point B_0 and then pick points $B_1, B_2, \ldots B_n$ such that for each $k = 1, 2, \ldots, n$ the line segments $A_{k-1}A_k$ and $B_{k-1}B_k$ have common mid points.

Show that $B_0 = B_n$ provided that the number of points n is even.